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Jñānābha ज्ञानाभ

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Special Volume to Honour
Professor J.N. Kapur
on his 70th Birthday

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1994-95

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This volume of
JÑĀNĀBHA
is being dedicated to honour
Professor J.N. Kapur
on his 70th Birthday

PROFESSOR JAGAT NARAIN KAPUR

Born : September 7, 1923

Felicitation on his 70th Birthday : September 7, 1993

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मनीषी, गणित गणपति, उदारमना
देवपुरुष डा० कपूर का अभिनन्दन

डा०बी०बी०लाल (डी. लिट्)
अवकाश प्राप्त प्राचार्य दयानन्द वैदिक महाविद्यालय, उरई
तथा
भू०पू० कुलपति बुन्देलखण्ड वि०वि० झाँसी
की ओर से अभिनन्दन

– 'ज्ञानाभ' परिवार के संरक्षक एवं 'ज्ञानाभ' का श्रीगणेश कराने के लिए प्रेरक भूमिका का निर्वाह करते हुए, मुझे, प्रस्तोता के रूप में सेवा का एवं इस शोध पत्रिका से संबद्ध होने का सौभाग्य मिला । 'ज्ञानाभ' की विश्वयात्रा की उपलब्धि की पृष्ठभूमि में श्रद्धेय महामना डा० कपूर का प्रशंस्य योग रहा जिसके लिए उनके प्रति 'ज्ञानाभ' परिवार के साथ मैं भी नतमस्तक एवं कृतार्थ होकर अपनी विनम्र कृतज्ञता सादर ज्ञापित करता हूँ ।

– विश्व धर्म सम्मेलन में शिकागो में महर्षि विवेकानन्द को बोलने के लिए 'शून्य' प्रस्तावित किया गया । 'शून्य', गणित का शून्य जो ब्रह्म विद्या, राजयोग और आत्म दर्शन की भी व्याख्या करता है, यह व्यवस्था भारत के प्रतिनिधि ने ही दी । इसी संदर्भ में मैंने डा० कपूर को 'मनीषी' एवं 'देवपुरुष' कहने का साहस किया है ।

– गणित एकाग्रता का ही पर्याय है तथा एकाग्रता समस्त योग और ब्रह्मविद्या की आधार शिला है । मेरी ऐसी मान्यता है कि गणितज्ञ अपने अध्ययन अनुशीलन विषय 'गणित' के ही कारण योगी की कोटि में आ जाता है । सारस्वत डा० कपूर एक 'गणितज्ञ' हैं, गणित पुरुष हैं तथा राजयोगी हैं, ब्रह्मविद् हैं । उनका अभिनन्दन भारतीय मनीषा का अभिनन्दन है, भारतीय दर्शन का अभिवादन है, भारतीय आत्मा के प्रति विनम्र नमन है ।

– हमारी कामना और परमपिता परमात्मा से हार्दिक प्रार्थना है कि यशस्वी डा० कपूर शतायु हों, स्वस्थ एवं प्रसन्न रहें । नयी पीढ़ी अपनी विकासोन्मुख दिशा में उनसे प्रेरणा प्राप्त करे, उनके, आशीर्वाद से अग्रसर हों तथा उनकी भावस्वरूपा छबि में उनकी उल्लेख्य ऊँचाइयों को प्राप्त करे ।

एवमस्तु ! एवमस्तु !! एवमस्तु !!!

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Jñānābha, Vol 24, 1994
(Dedicated to Professor J.N. Kapur on his 70th Birthday)

PROFESSOR J.N. KAPUR : MAN AND MATHEMATICIAN

By

R.C. Singh Chandel

Secretary, Vijnana Parishad of India,
D.V. Postgraduate College, Orai - 285001 U.P., India

Professor J.N. Kapur (Jagat Narain Kapur), today needs no introduction to the academic community of India. However, to make formality we may say that simple, unassuming, energetic, dynamic, witty and over mailing, a varasious reader, an aloquent speaker, a prolific writer, an out standing researcher, an excellant teacher, an able administrator, an eminent educationist, encouraging his followers, a true and devoted salesman of Mathematics; these are only some of the qualities of Professor Kapur as a man.

Professor Kapur was born in Delhi on September 7, 1923. He was educated at D.A.V. High School, Daryaganj and at Hindu College, University of Delhi. He had a first class first throughout. He beat the Delhi University records in B.A. (Hons) and M.A. examinations by wide margins securing 91% and 97% marks respectively. He also won three gold medals for his brilliant performance.

Soon after passing his M.A., Professor Kapur joined Hindu College Delhi as a Lecturer, where he served for fifteen years and established himself as a teacher of repute. In 1957, he got Ph. D. from Delhi University. In 1959 he joined the University of Delhi as a Reader. In 1961 he left Delhi to join I.I.T. , to organise Mathematics Department. His stay at Kanpur (1961- 1970) was a period of intense research activity. By his imagination, foresight, understanding and hard work, he established to Department at I.I.T. as one of the strongest schools of Mathematics in the Country. On 31st December, 1970, Professor Kapur took up an assignment as Vice-Chancellor of Meerut University Meerut at so early an age of forty seven. He produced himself as good administrator as a mathematician. During the period of his Vice-Chancellorship, he visited all the 56 Colleges affiliated to the Meerut University many times and talked with students, teachers, managements and public to solve their problems. He made the atmosphere of the University peaceful and brought the University to the top. It was due to his serious efforts that the right kind of academic atmosphere prevailed and the academic activities of the University were consequently at their peak during the period he was the Vice-Chancellor.

During his Vice-Chancellorship, he continued his academic work also with same sprite and when he found that his administrative duties were interfering with his academic work, he resigned during

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the second term of Vice-Chancellorship of Meerut University and Joined back IIT, Kanpur in August, 1974.

In September, 1983 his 60th birthday was celebrated in different part of India. He was retired but was requested to continue for three years more at I.I.T., Kanpur. He was only the Professor of the presetigious Institutie to have been given this honour.

In October, 1986 he returned to Delhi and joined I.I.T., Delhi, Delhi University as an Honourary Visiting Professor for a period of two years.

He was also selected as an Indian National Science Academy (INSA) Senior Scientist for 1987, 1988 and 1989. In October 1988, he joined the School of Computer and System Sciences of Jawahar Lal Nehru University, New Delhi as Honourary Professor for life. Associated with Professor Kapur since 1970 through "JANABHA" a Journal of "VIJNANA PARISHAD OF INDIA" whose I am one of the Editors and Prof. Kapur is its Chief Advisor. He always en courages us. The "VIJNAN PARISHAD OF INDIA" in its present form is due to inspiration of Professor Kapur. He may be treated as founder President of present "VIJNANA PARISHAD OF INDIA" because before him Parishad was an unregistered body having no President.

There are millions of students who appear in university examination. Out of these, there are a few thousands who get first division in all positions in all their examinations. Out of these, three may be a few who beat earlier university records in both their first and second degree examinations. Out of these there is only one who beat these earlier records by wide margins of 4% and 15% marks. This man is Professor Kapur.

There are tens of thousand of persons who get Ph.D. degrees. A number of their theses are based on ten or more published papers. There was one thesis of more than 900 pages submitted for Ph.D. degree and this was based on 30 published papers. This was submitted by Prof. Kapur.

There are tens of thousand of mathematicians who publish research papers. Out of these, a few hundred publish more than 100 research papers. Prof. Kapur is one of them, having published more that 400 research papers.

There are research workers who work in one area all their life. Some work in two or three areas. Very few worked in ten or more different areas. Prof. Kapur is one of them.

There are many research guides who gude students for Ph.D. degree. There are some who guide more than 30 students. Prof. Kapur is one of them.

There are many persons who write general scholarly, articles on educational matters. There are a few who publish more than 500 such articles. Prof. Kapur is one of them.

There are amny authors who write dozens of non-ficion books. Prof. Kapur is one of them, having written more than 60 books, more than 50 of which are innovative in nature.

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There are a large number of authors whose books are published locally. There is a smaller number of authors whose books are published by international publishers. Prof. Kapur is one of them. Four of his books have already been published in USA and Canada.

There are persons who have written books for primary level or secondary level or senior secondary level or undergraduate level or post graduate level or reserach level or general books or books on education. Prof. Kapur has written books at all these levels.

There are tens of thousands of scientists in the country. Out of these about a thousand are Fellows of one or the other of the three National Science Academies. There are about 200 who are Fellows of all the three Academies. Proff. kapur is one of them and is quite senior among them.

There are many mathematicians who have been president of Indian mathematical Society or Calcutta Mathematical Society or Bharat Ganita Parishad or Indian Society of Theoretical and Applied Mechanics or indian Society of Agricultural Statistics or Mathematical Section of Indian Science Congress or Physical Science Section of National Academy of Sciences or of Associations of mathematics Teachers of India or of Vijnana Parishad of India or of Mathematical Association of India or of Indian Society of Industrial and Applied Mathematics. Only Prof. Kapur has been the president of all of them.

He has also been Vice-President of Operation Research Society of India, Indian Society for History of Mathematics, and Society of Scientific Values. This also reflects his wide interest in Pure and Applied Mathematics, Statics Theoretical and Applied Mechanics, Operations Research, Mathematics Education, History of Mathematics and Scientific Values.

There are many Indian mathematicians who have been visiting professors abroad and out of them, some have been visiting professors in developed countries like USA, Canada and Australia. Prof. Kapur has been a full visisitng professor in eight universities of developed countries. He has been a visiting professor not only in departments of Pure and Applied Mathematics but also in departments of Management Science, Industrial Engineering, School of Business and System Engineering. At present he is honorary professor in a school of Computer and System Sciences. This also shows the wide range of his interests.

There are many persons who worked as editors of journals or on editorial boards of journals. Prof. Kapur has worked as editor of four journals and has been on the editorial board of two dozen other journals.

Many persons have directed summer schools organised by the UGC or NCERT. Some have directed a number of them. Prof. Kapur has directed about 30 of them. What is the more important is that he is one of few persons who have organised the summer schools on their own initiative.

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There have been thousands of Vice-Chancellors of Indian Universities, but only a few who got this position before they were even 47 years old and fewer still who were requested to accept this position. Infact his Cancellor got levave of absence for him from IIT/K without his even asking for it. He was a Vice Chancellor who stood firmly for 100% fairness and impartiality in all admissions, appointments and examinations. It was he who conducted the CPMT examination in UP for four years without a single complaint. There were plenty of complaints before and after his time, but none during his time. In fact in an extraodinary resolution, the UP government congratulated him on his outstanding work which looked like a miracle at that time.

Again many eminent persons like their birthdays to be celebrated. However, when his students collected some funds for celebrating his 50th birthday, he prohibited them from doing so and these funds became the nucleus for a mathematical Sciences Trust Society to work for the development of Mathematical Sciences in India.

When the UP government wanted to give honararium for the work he did for CPMT, he refused to accept it and at his instance the UP governement donated the amount to MSTS. He gave the royality of most of his books to the Trust. The Trust has published about 30 books so far and gives prizes worth about Rs. 20,000 per year to mathematical Olympial winners and to all those who stand first in Mathematics in Board and University examinations all over the country.

Many persons can claim to have done something for mathematics educations research by using government money. He does not use government money. He believes in giving and not in taking. He believes that the country will develop when the people depend on their own resources rather than on government resources for serving the country.

Many persons in India have personal libraries. Some mathematicians and Scientists in India have their personal libriaries. But few have a library of the size of Prof. Kapur's Library. His library consists of more than 5000 books, 1000 journals and 200 Ph.D. theses.

Prof. Kapur has won many awards including GP Chatterji award of ISCA, Education Minister's Gold medal of N.A.S.C., Yogiji Maharaj Ceneterary Award and best paper awards in many international conferences. He has won honours including Visiting Professorships, Fellowships of Academies, Presidentships of Societies, Vice- Chancellership etc. but he has never asked for them. He has never hinted to anybody that he should be considered for any one of these.

He believes in the joy of work. He enjoys reading, writing, lecturing, severing. He neither expects appereciation or reward for his work. If there is anything he expects from his students. Friends and admires, it is only that they may give their best to mathematics science country and man-kind.

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Prof. Kapur has many firsts to his credit. He was the first to organise the summer schools in India, of course with the cooperation of others. He was the first head of the mathematics department of IIT, Kanpur and he built this department to international stadards in a short time. He was the first author to write books on Mathematical Statics, Mathematical Modelling, Mathematical Models in Biology and Medicine, Maximum Entropy Models and Entropy Optimisation Priciples, Biographies of Indian Mathematicians and Olympiad Problems. In many cases his books were the first of their kind in the world.

PROFESSOR J.N. KAPUR AT A GALANCE

NEME	Jagat Narain Kapur
DATE OF BIRTH	7-9-1923
AFFILIATIONS	Honorary Professor School of Computer and Systems Sciences, Jawahar Lal Nehru University, New Delhi - 110067 & Honorary Director Mathematical Science Trust Society, New Delhi
ADDRESS	C-766, New Friends Colony, New Delhi, 110065 India (001) 6832290.
EDUCATION	B.A. (Hons.) Mathematics, 1942, Delhi University, India **First Class First** M.A. Mathematics, 1944, Delhi University, India **First Class First** Ph.D. Mathematics, 1957, Delhi University, India Certificate course in Statistics (ICAR) **First Class First** in first batch. **Broke Previous Delhi University Records** in both B.A. (Hons.) and M.A. examinations by margins of 4 percent and 15 percent marks by securing 91 and 97 percent marks respectively. **Won Three Gold Medals and Six Scholorships.**
POSITIONS HELD	
1944-59	Senior Lecturer, Hindu College (Graduate School) Delhi University, India 1945-1959
1959-1961	Reader (Associate Professor) Institute of Postgraduate Studies, Delhi University, India. 1959-1961
1961-86	Professor, Indian Institute of Technology, Kanpur, India 1961- 1986.
1986-88	Visiting Professor , Mathematics Department, IIT, Delhi & Delhi University,

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1987-89	Senior Scientist, Indian National Science Academy,
1988-90	Adjunct Professor, Waterloo University Canada
1988 upto date Honly.	Professor, school of Computer and Systems Science, Jawahar Lal Nehru University, New Delhi.

VISITING PROFESSOR

University	Country	Year	Department
Arkansas	USA	1969	Mathematics
Carnegie-Mellon	USA	1969-70	Mathematics
Siena	Italy	1970	Mathematics
Manitoba	Canada	1980-81	Business and Acturial Mathematics
Waterloo	Canada	1981-82	Applied Mathematics
Waterloo	Canada	1982	System Engineering
Manitoba	Canada	1983	Actuarial and Management Sciences
Manitoba	Canada	1984	Industrial Engineering
New South Wales	Australia	1984	Applied Mathematics
Flinders	Australia	1984	Mathematical Sciences
Carleton	Canada	1985	Business School
Waterloo	Canada	1985	Systems Design Enginnering
Waterloo	Canada	1986	System Design Engineering
I.I.T. Delhi	India	986-88	Mathematics
Waterloo	Canada	1987	System Design Engineering
Delhi	India	1987-88	Mathematics
Waterloo	Canada	1988-89	Systems Design Engineering
Waterloo	Canada	1990, 91 & 1992	Systems Design Engineering

FELLOWSHIPS IN PROFESSIONAL ORGANISATIONS

		YEAR OF ELECTIONS
Indian Academy of Science	F.A.Sc.	1965
national Academy of Sciences	F.N.A.Sc.	1965
Institute of Mathematics and its Applications, UK	F.I.M.A.	1966
Indian National Science Academy	F.N.A.	1969

PRESIDENTSHIP OF PROFESSIONAL ORGANISATIONS

Indian Science Congress Association (Mathematics Section)	1968
Bharat Ganita Parishad	1969

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Indian Mathematical Society	1971,1972
Calcutta Mathematical Society	1975
Association of Mathematics Teachers of India	1977-90
Indian Society of Theroretical and Applied Mechanics	1978
Mathematical Association of India	1979*
Indian Society of Agricultural Statistics	1984
National Academy of Sciences (Physical Sciences Sect.)	1984
Mathematical Sciences Trust Society	1973*
Vijnana Parishad of India	1985*
Indian Society of Industrial and Applied Mathematics	1993*
Indian National Commission on History of Sciences (Modern Period)	1993

CHIEF EDITOR

The Mathematics Seminar	1963-68
Bulletin, Mathematical Association of India	1969*
The Mthematics Student (IMS)	1968-70
The Mathematical Education (UGC)	1984*
National Council of Educational Research and Training Upper Primary mathematics Text Books	1987-89
Indira Gandhi National Open University Text books on Mathematics	1987-89

MEMBERSHIP (PAST OR CURRENT)

- of Editorial Boards of Twenty Indian and Foreign Journals
- (Chief Advisor : Jnānābha Published by Vijnana Parishad of India, since 1970)
- of 25 professional societies in the worded
- International Commission of Mathematics Instruction (India's Representative)
- Indian National Commission on History of Sciences
- Excutive Committee and Council, Indian Science Congress Association.
- Council, Sectional Committee and Basic Sciences Committee of INSA
- Council, of National Academy of Sciences
- Organizing Committee of Fourth International Conference on Mathematical Modelling.
- Organizing Committee of First and Second International Congress of Teaching of Mathematical Modelling.
- International Conference on Transporation (Chairmann, Mathematics Committee)

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- Central Board of Secondary Education (Chairman, Mathematics Committee)
- national Council of Teacher Education
- university Grants Commission Mathematics Panel
- Innovations Committee of National Council of Educational Research and Training, India
- Review Committee of Regional Colleges of Education (Chairman)
- Council of Indian Society of History of Mathematics (Vce-President)
- Council of Operations Research Society of India (VIce-President)
- Science Education Forum of Indian Sceince Congress (Convenor)
- Parmar Institute of Mathematial Sciences Shimla, India (Honorary Dicector)
- Indian Mathematical Society (Academic Secretary')
- Council of Society of Scientific Values (Vice-Chairman)
- National Commiitee on Mathematical Sciences (DST) (Chairman)
- National Committee on Mathematical Education and Research (DST)
- Governing bodies of nine U.P. Engineering Colleges.

PUBLICATIONS : RESEARCH PAPERS (ABOUT 4000 PUBLISHED PAGES)

Internal Ballistics of Orthodox Guns (21), Internal Ballistics of Special Guns and Rockets (15), Form Function for Multitubular Charge (8), International Ballistics of Cpmposite and Moderated Charges (14), General Fluid Flows (15), Compressible Fluid Flows (9), General Non-Newtonian Fluid Flows (27), Non-Newtonian Fluids in Inlet Regions (14), Conducting Non-Newtonian Fluid Flows (3), Megneto Hydrodynamics (20), Heat Transfer (2), General Ppulation Dynamics (8), Difference Equation Population Models (6), Population Models with Time Delays (8), Prey Predator and Competition Models (7), Age-Structured Population Models (16), Mathematical Bioeconomics (14) Bio-mechanics (4), Compartment Analysis (3), Stochastic Processes (14), Measures of Information and Their Properties (50), Entropy Optimization Principles (16), Maximum Entropy Principle in Statistics Statistical Mechanics, and Operations Research (19). MEP Models in Marketing, Political Science, Economics Business, Search Theory, Population Dynamics, Pattern Recognition, Image processing, Fexible Manufacturing Systems, Coding Theory and Regional and Urban Planning (16), General Information Theory (9), Financial Mathematics Decision Theory and Social Sciences (7), Flexible Manufcaturing Systems (12), Fibonacci Numbers (12). Geometry (23), Innovation Deffusion Models (2) Survey Papers (9), Miscellaneous (10), Measures of Information and Their Applications (60), Statistics (4).

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PUBLICATIONS : GENERAL ARTICLES (ABOUT 10000 PAGES)

Mathematical Education (178), Expository Mathematics (135), Applications of Mathematics (55), Mathematics Problem (12), Education (22) Higher Education (86), Vice-Chancellor Addresses (34), Convocation Address (6), Articles/Book Reviews (115), General Articles (18), Editorial (30), Scientific Values (6).

PUBLICATIONS BOOKS : (ABOUT 16,000 PAGES)

Advanced Level Books (13), Expository Mathematics (23), Mathematics Olympiad Problem Solving (6), Text Books (12), Mathematics Education (17), Higher Edication (3), Book Edited (10).

ADMINISTRATIVE EXPERIENCE

Vice-Chancellor, Meerut University, India (1971-74). Head of postgraduate mathematics Deptt. (32 Years), Acting Director, Indian Institute of Technology, Kanpur, India on different occasions for approximately 250 days Director of Thirty Summer Schools of Mathematics, Organizer of a dozen national and international Conferences.

AWARDS

National Academy Gold Medal for best research	1980
National Lecturer, University Gransts Commission	1982-83
G.P. Chatterjee Award, Indian Science Congress Association	1988
Distinguished Service Award, Mathematical Association of India	1983
Platinum Jubilee Lectureship, Indian Science Congress Association	1988
Best paper Award, Administrative Science Congress Association	1986
Best paper Award, Administrative Science Association, Canada	1986
Felicitation by Yogiji Maharaj Trust, Gandhigram	1992

UNIVERSITIES VISITED

USA (39), Canada (12), Italy (2), UK (12), West Germany (9), Australia (12), Netherlands (3), Singapore (1), Iran (2), Thailand (1), Bangladesh (1).

MATHEMATICS EDUCATION PROJECTS VISITED

USA (14), UK (8), Netherlands (2), Italy (3), Australia (8).

SEMINARS/LECTURES

Given in India (about (500), Given in other countries (about 200)

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(Dedicated to Professor J.N. Kapur on his 70th Birthday)

AN OPERATIONAL CALCULUS FOR THE INDEX $_2F_1$ – TRANSFORM

By

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ABSTRACT

In this paper an operational calculus for the index $_2F_1$ – transform is developed. Furthermore, this calculus is applied to solve certain type of differential equations involving generalized functions.

1. Introduction

The index $_2F_1$-transform for real-valued functions is defined in [3] (see also [4], by the formula:

$$F(\tau) = \int_0^\infty \boldsymbol{F}(\mu, \alpha, \tau, x) f(x) dx, \qquad \text{... (1.1)}$$

where

$$\boldsymbol{F}(\mu, \alpha\, \tau, x) = {}_2F_1(\mu + \tfrac{1}{2} + i\tau, \mu + \tfrac{1}{2} - i\tau; \mu + 1; -x) x^\alpha, \qquad \text{... (1.2)}$$

$_2F_1$ being the Gauss hypergeometric function. Here α and μ are complex parameters and τ is a positive real parameter.

This integral transform was extended to certain spaces of generalized functions (see [5]). For it, a testing function space (namely, $U_{\alpha, \mu, \alpha}$) was introduced in the following manner: a smooth function $\phi(x)$ on $0 < x < \infty$, belongs to $U_{a, \mu, \alpha}$ if and only if

$$\gamma_{k, a, \mu, \alpha}(\phi) = \sup_{0 < x < \infty} \left| (2x+1)^a x^{\mu/2 - \alpha} (x+1)^{\mu/2} A_x^k \phi(x) \right| < \infty \qquad \text{... (1.3)}$$

with $a \in [0, \frac{1}{2})$, $\mu, \alpha \in C$, $k = 0, 1, 2, \ldots$ and where A_x denotes the differential operator :

$$A_x := x^{\alpha - \mu}(x+1)^{-\mu} D_x x^{\mu+1} (x+1)^{\mu+1} D_x x^{-\alpha}. \qquad \text{... (1.4)}$$

The topology in $U_{\alpha, \mu, \alpha}$ is induced by the countable family of seminorms $\{\gamma_{k, a, \mu, \alpha}\}$. It turns out that the kernel function $\boldsymbol{F}(\mu, \alpha, \tau, x)$ is in $U_{\alpha, \mu, \alpha}$. As it is usual, the dual space of $U_{a, \mu, \alpha}$ is denoted by $U'_{a, \mu, \alpha}$, and I denotes the real interval $(0, \infty)$.

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The transform $F(\tau)$ of the generalized function $f \in U'_{a,\mu,\alpha}$ is then defined as follows :

$$ {}_2\mathcal{F}_1(f) = F(\tau) = \langle f(x), \boldsymbol{F}(\mu, \alpha, \tau, x) \rangle , \quad \tau > 0 \qquad \ldots (1.5) $$

and for $a \in [0, \frac{1}{2})$ and $f \in \mathcal{E}'(\boldsymbol{I})$ the following inversion formula holds :

$$ \langle f, \phi \rangle = \lim_{N \to \infty} \left\langle \int_0^N S(\mu, \tau)\, \boldsymbol{G}(\mu, \alpha, \tau, x)\, F(\tau) d\tau, \phi(x) \right\rangle \qquad \ldots (1.6) $$

with $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \mu > 0$, $\frac{1}{8} < \operatorname{Re}(\mu - \alpha) < \frac{1}{4}$ and $\operatorname{Re}(\frac{\mu}{2} - \alpha) < -\frac{1}{2}$. Here,

$$ S(\mu, \tau) = \frac{2}{\pi \Gamma(\mu + 1)^2} \tau \, sh\, \pi\tau \, \Gamma(\mu + \tfrac{1}{2} + i\tau)\, \Gamma(\mu + \tfrac{1}{2} - i\tau) $$

and

$$ \mathbf{G}(\mu, \alpha, \tau, x) = x^{\mu - \alpha}\, {}_2F_1(\tfrac{1}{2} + i\tau, \tfrac{1}{2} - i\tau ; \mu + 1; -x). $$

Further more, the following uniqueness theorem was proved : if $f, g \in \mathcal{E}'(\boldsymbol{I})$ and ${}_2\mathcal{F}_1[f] = {}_2\mathcal{F}_1[g]$, then $f = g$.

In [5] was also established that for every $f \in U'_{a,\mu,\alpha}$ there exists a non- negative integer τ such that

$$ {}_2\mathcal{F}_1(f) = F(\tau) = O\left(\tau^{2r - \operatorname{Re}\mu - \frac{1}{2}}\right), \quad \tau \to \infty \qquad \ldots (1.7) $$

On the other hand, the following relation holds :

$$ {}_2\mathcal{F}_1\left((A_x')^{\kappa} f\right) = (-1)^{\kappa} \left[\left(\mu + \tfrac{1}{2}\right)^2 + \tau^2\right]^{\kappa} {}_2\mathcal{F}_1(f), \quad \kappa \in N. \qquad \ldots (1.8) $$

A'_x being the adjoint operator of A_x defined by (1.4).

In this paper we consider the operational equation :

$$ P(A'_x)u = g, $$

where $g \in \mathcal{E}'(\boldsymbol{I})$, P is any polynomial different from zero in $(-\infty, 0)$ and $\alpha, \mu \in \boldsymbol{R}$.

Our aim is to find a generalized function $u \in \mathcal{E}'(\boldsymbol{I})$ satisfying the above operational equation. Spaces $\mathcal{D}(I)$ and $\mathcal{E}(I)$ and their duals $D'(I)$ and $\mathcal{E}'(I)$have its usual meaning [7].

2. The operational equation

Let us consider the following equation :

$$ P(A'_x)u = g, \qquad \ldots (2.1) $$

where $g \in \mathcal{E}'(\boldsymbol{I})$, $P(z)$ denotes an arbitrary polynomial without zeros in $-\infty < z \leq 0$ and just as in (1.8) A_x' is the differential operator

$$ A_x' = x^{-\alpha} D_x x^{\mu+1} (x+1)^{\mu+1} D_x x^{\alpha-\mu} (x+1)^{-\mu}, $$

which is the adjoint of A_x.

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By applying the generalized index ${}_2\mathcal{F}_1$-transform to both sides of (2.1) and using (1.8),we get

$$P\left(-\left(\mu+\tfrac{1}{2}\right)^2-\tau^2\right)U(\tau)=G(\tau),$$

where $U(\tau)$ and $G(\tau)$ represent the generalized index ${}_2\mathcal{F}_1$-tranform of $u(x)$ and $g(x)$, respectively. Now, since $P\left(-\left(\mu+\tfrac{1}{2}\right)^2-\tau^2\right)\neq 0$,by means of the inversion formula (1.6), and for any $\phi\in\mathcal{D}(I)$ it is found that

$$\langle u,\phi\rangle = \lim_{N\to\infty}\left\langle \int_0^N S(\mu,\tau)\, G(\mu,\alpha,\tau,x)\,\frac{G(\tau)}{P\left(-\left(\mu+\tfrac{1}{2}\right)^2-\tau^2\right)}\,d\tau,\ \phi(x)\right\rangle \qquad \ldots (2.2)$$

Thus, we have obtained formally a solution of (2.1). Now we must prove that (2.2) is certainly a generalized function and that it satisfies the equation (2.1). For this, the following Lemma is required.

Lemma 2.1 *For $x\in[a,b]$ and $\mu>0$, there exists a $T_1>0$ such that $\forall\ \tau\le T_1$*

$$\left|{}_2F_1\left(\tfrac{1}{2}+i\tau,\tfrac{1}{2}-i\tau\,;\mu+1;-x\right)\right|\le B_1$$

and there also exists a $T_2>0$, such that $\forall\ \tau\ge T_2$

$$\left|{}_2F_1\left(\tfrac{1}{2}+i\tau,\tfrac{1}{2}-i\tau\,;\mu+1;-x\right)\right|\le B_2\,\tau^{-\frac{1}{2}}$$

with $B_1>0$ *and* $B_2>0$.

Proof. Starting from the integral representation [6] (p. 248).

$${}_2F_1(\alpha,\beta;\gamma;-x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\,\frac{4}{x^{\alpha+\beta/2}}\int_0^\infty K_{\alpha-\beta}\left(\frac{2s}{\sqrt{x}}\right)J_{\gamma-1}(2s)s^{\alpha+\beta-\gamma}\,ds.$$

which is valid for $\mathrm{Re}\,\alpha>0$, $\mathrm{Re}\,\beta>0$, and K, J being the well-known Bessel functions, we can write :

$${}_2F_1\left(\tfrac{1}{2}+i\tau,\tfrac{1}{2}-i\tau;\mu+1,-x\right)=\frac{\Gamma(\mu+1)}{\Gamma(\frac{1}{2}+i\tau)\Gamma(\frac{1}{2}-i\tau)}\,4x^{-1/2}\int_0^\infty K_{2\tau i}\left(\frac{2s}{\sqrt{x}}\right)J_\mu(2s)s^{-\mu}ds$$

Hence, for a suitable $T_1>0$, if $\tau\le T_1$ one has

$$\left|K_{2\tau i}\left(\frac{2s}{\sqrt{x}}\right)\right|\le K_0\left(\frac{2s}{\sqrt{x}}\right).$$

Thus, for $\tau\le T_1$,

$$\left|{}_2F_1\left(\tfrac{1}{2}+i\tau,\tfrac{1}{2}-i\tau;\mu+1;-x\right|\le B_1$$

with $B_1>0$.

On the other hand, for $\tau\to\infty$,

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$$\left| K_{2\tau i}\left(\frac{2s}{\sqrt{x}}\right)\right| \leq M_1\, e^{-\pi\tau}\, \tau^{-\frac{1}{2}}$$

holds for certain $M_1 > 0$ (see [2] 7.14.2 (69)).

Moreover, $\Gamma\left(\frac{1}{2}+i\tau\right)\Gamma\left(\frac{1}{2}-i\tau\right) = \frac{\pi}{ch\,\pi\tau}$.

Then for $\tau \geq T_2$:

$$\left|{}_2F_1\left(\tfrac{1}{2}+i\tau, \tfrac{1}{2}-i\tau; \mu+1; -x\right)\right| \leq M_2\tau^{-1/2}e^{-\pi\tau}\, ch\pi\tau \int_0^\infty s^{-\mu}\, |J_\mu(2s)|\, ds \leq B_2\tau^{-\frac{1}{2}}$$

B_2 being a positive constant □

Now, if we again consider the equation (2.1) it results, after taking a polynomial $Q(z)$ of degree $r+\mu+1$ without zeros on $-\infty < z \leq 0$, r being the integer given in (1.7), that the convergence of the right- hand side of (2.2) can be established as follows. First, we have

$$\left\langle \int_0^N S(\mu,\tau)\, \boldsymbol{G}(\mu, \alpha, \tau, x)\, \frac{G(\tau)}{P\left(-(\mu+\frac{1}{2})^2 - \tau^2\right)}\, d\tau,\ \phi(x) \right\rangle =$$

$$\left\langle Q\left(A'_x\right) \int_0^N \frac{S(\mu,\tau)\boldsymbol{G}(\mu, \alpha, \tau, x)G(\tau)}{P\left(-(\mu+\frac{1}{2})^2 - \tau^2\right) Q\left(-(\mu+\frac{1}{2})^2 - \tau^2\right)}\, d\tau,\ \phi(x) \right\rangle =$$

$$\left\langle \int_0^N \frac{S(\mu,\tau)\, \boldsymbol{G}\,(\mu, \mu, \tau, x)\, G\,(\tau)}{P\left(-(\mu+\frac{1}{2})^2 - \tau^2\right) Q\left(-(\mu+\frac{1}{2})^2 - \tau^2\right)}\, d\tau,\ Q\,(A_x)\, \phi(x) \right\rangle .$$

This follows by making use of :

$$A'_x\, \boldsymbol{G}\,(\mu, \alpha, \tau, x) = -\left[(\mu+\tfrac{1}{2})^2 + \tau^2\right] \boldsymbol{G}\,(\mu, \alpha, \tau, x).$$

Thus, if the support of ϕ is contained in $[a, b]$, the expression (2.2) can be written as

$$\lim_{N\to\infty} \int_0^N \frac{S(\mu,\tau)G(\tau)}{P\left(-(\mu+\frac{1}{2})^2 - \tau^2\right) Q\left(-(\mu+\frac{1}{2})^2 - \tau^2\right)}\, d\tau \int_a^b \mathbf{G}(\mu, \alpha, \tau, x)\, \phi(x)\, dx \qquad \ldots (2.3)$$

Then, by invoking Lemma 2.1 we can find suitable constants C, D, E, N_1 and N_2 such that

$$\int_0^N \left| \frac{S(\mu,\tau)G(\tau)}{P\left(-(\mu+\frac{1}{2})^2 - \tau^2\right) Q\left(-(\mu+\frac{1}{2})^2 - \tau^2\right)} \right| d\tau \int_a^b |\boldsymbol{G}\,(\mu\,, \alpha, \tau, x)\, \phi(x)|\, dx \leq$$

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$$\leq C \int_0^{N_1} \left| \frac{S(\mu, \tau)}{P\,(-(\mu+\frac{1}{2})^2 - \tau^2)\; Q\,(-(\mu+\frac{1}{2})^2 - \tau^2)} \right| d\tau +$$

$$+ D \int_{N_1}^{N_2} \left| \frac{G(\tau) S(\mu, \tau)}{P\,(-(\mu+\frac{1}{2})^2 - \tau^2)\; Q\,(-(\mu+\frac{1}{2})^2 - \tau^2)} \right| d\tau.$$

$$\int_a^b \left| {}_2F_1\left(\frac{1}{2} + i\tau, \frac{1}{2} - i\tau\,;\, \mu+1;\, -x\right) \right| dx +$$

$$E \int_{N_2}^{N} \frac{\tau^{2\tau - \frac{1}{2}} \left| S(\mu, \tau) \right|}{\left| P\,(-(\mu+\frac{1}{2})^2 - \tau^2)\; Q\,(-(\mu+1/2)^2 - \tau^2) \right|} d\tau.$$

Now, it is not difficult to prove the boundedness of the first and the second integrals above. For the third one, taking into account that , for $\tau \to \infty$,

$$| S(\mu, \tau) | \leq M \tau^{2\mu+1}, \quad M > 0,$$

this integral converges as $N \to \infty$.

Therefore,the integral (2.3) exists and thus, by the completeness of $\mathcal{D}'(I)$, there exists $f \in \mathcal{D}'(I)$ such that

$$\lim_{N \to \infty} \left\langle \int_0^N S(\mu, \tau)\, \mathbf{G}(\mu, \alpha, \tau, x) \frac{G(\tau)}{P\,(-(\mu+\frac{1}{2})^2 - \tau^2)} d\tau,\, \phi(x) \right\rangle = \langle f, \phi \rangle.$$

... (2.4)

The generalized function f determined in (2.4) is the restriction of $u \in \mathcal{E}'(I)$ to $\mathcal{D}'(I)$. In view of the continuity of the differentiation and the multiplication by x and by $\frac{1}{x}$ in $\mathcal{D}'(I)$, one can show that, for any $\phi \in \mathcal{D}(I)$

$$\lim_{N \to \infty} \left\langle P(A'_x) \int_0^N S(\mu, \tau)\, \mathbf{G}(\mu, \alpha, \tau, x) \frac{G(\tau)}{P\,(-(\mu+\frac{1}{2})^2 - \tau^2)} d\tau,\, \phi(x) \right\rangle = \langle P(A'_x) f, \phi \rangle$$

and from this expression and the above inversion formula, it finally follows that

$$\langle g, \phi \rangle = \langle P(A'_x) f, \phi \rangle$$

This result proves that the generalizd function $f \in \mathcal{D}'(I)$ is the restriction of $u \in \mathcal{E}'(I)$ to $\mathcal{D}(I)$ and that it satisfies the equation (2.1)

Acknowledgements

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MULTIDIMENSIONAL FRACTIONAL DERIVATIVES OF THE MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT

In the present paper, multidimensional fractional derivatives of the multiple hypergeometric functions of several variables have been obtained.

1. INTRODUCTION

Motivated by the earlier work of Srivastava and Goyal [11], Srivastava, Chandel and Vishwarkarma [12] derived various fractional derivatives involving the generalized multiple hypergeometric function of Srivastava and Daoust [10] and the multivariable H-function of Srivastava and Panda [13]. They also discussed the special cases of these results involving the multiple hypergeometric functions of several variables and their confluent forms defined by Lauricella [8], Exton [5],[6]), Chandel [1], Chandel and Gupta [2], and Karlsson [7].

In the present paper, we apply the same techniques in order to derive the **multidimensional** fractional derivatives involving the multiple hypergeometric functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ of Lauricella [8] and their confluent forms $\Xi_1^{(n)}$, $\Phi_2^{(n)}$, $\Psi_2^{(n)}$, $\Phi_3^{(n)}$, ${}^{(k)}_{(1)}E_D^{(n)}$, ${}^{(k)}_{(2)}E_D^{(n)}$, of Exton ([5], [6]) ${}^{(k)}_{(1)}E_C^{(n)}$ of Chandel [1], ${}^{(k)}F_{AC}^{(n)}$, ${}^{(k)}F_{BD}^{(n)}$, ${}^{(k)}F_{AD}^{(n)}$ of Chandel and Gupta [2], and their confluent forms ${}^{(k)}_{(1)}\Phi_{AC}^{(n)}$, ${}^{(k)}_{(2)}\Phi_{AC}^{(n)}$, ${}^{(k)}_{(1)}\Phi_{AD}^{(n)}$, ${}^{(k)}_{(1)}\Phi_{BD}^{(n)}$ and ${}^{(k)}F_{CD}^{(n)}$ of Karlsson [7].

2. FRACTIONAL DERIVATIVES

Making an appeal to the formula [9,p.67]

$$D_x^{\mu}\{x^{\lambda}\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\,x^{\lambda-\mu},\ \mathrm{Re}(\lambda) > -1, \tag{2.1}$$

we derive the folloing fractional derivatives involving the above multiple hypergeometric functions :

$$D_{x_1}^{\lambda_1-\mu_1}\ldots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\ F_A^{(n)}[a, \mu_1, \ldots, \mu_n; \right. \tag{2.2}$$

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$$c_1, \ldots, c_n; z_1x_1, \ldots, z_nx_n]\Big\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} F_A^{(n)} [a, \lambda_1, \ldots, \lambda_n ; c_1, \ldots, c_n ; z_1x_1, \ldots z_nx_n],$$

$$|z_1x_1| + \ldots + |z_nx_n| < 1, \ \mathrm{Re}(\lambda_i) > 0 , \quad i = 1, \ldots, n.$$

$$(2.3) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} F_A^{(n)} [a, b_1, \ldots, b_n ; \lambda_1, \ldots \lambda_n ; \right.$$

$$\left. z_1x_1, \ldots, z_nx_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} F_A^{(n)} [a, b_1, \ldots, b_n; \mu_1, \ldots, \mu_n ; z_1x_1, \ldots, z_nx_n],$$

$$|z_1x_1| \ 1 \ldots + |z_nx_n| < 1, \ \mathrm{Re}(\lambda_i) > 0, \quad i = 1, \ldots, n.$$

$$(2.4) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} F_B^{(n)} [\mu_1, \ldots, \mu_n, b_1, \ldots, b_n ; \right.$$

$$\left. c; z_1x_1, \ldots, z_nx_n] \right\}$$

$$= \prod_{J=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} F_B^{(n)} [\lambda_1, \ldots, \lambda_n, b_1, \ldots, b_n; c; z_1x_1, \ldots, z_nx_n],$$

$$max \left\{ |z_1x_1|, \ldots, |z_nx_n| \right\} < 1, \ \mathrm{Re}(\lambda_j) > 0, \ j = 1, \ldots, n.$$

$$(2.5) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} . F_B^{(n)} [a_1, \ldots, a_n, \right.$$

$$\left. \mu_1, .., \mu_n; c; z_1x_1, \ldots, z_nx_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} F_B^{(n)} [a_1, \ldots a_n, \ \lambda_1, \ldots, \lambda_n; c; z_1x_1, \ldots, z_nx_n],$$

$$max \left\{ |z_1x_1|^{1/2}, \ldots, |z_nx_n|^{1/2} \right\} < 1, \quad \mathrm{Re}\ (\lambda_i) > 0, i = 1, \ldots, n.$$

$$(2.6) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} F_C^{(n)} [a, b; \lambda_1, \ldots, \lambda_n; z_1x_1, \ldots, z_nx_n] \right\}$$

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$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} F_C^{(n)} [a, b; \mu_1, ..., \mu_n; z_1 x_1, ..., z_n x_n],$$

$$|z_1 x_1|^{1/2} + ... + |z_n x_n|^{1/2} < 1, \ \mathrm{Re}(\lambda_i) > 0, \ i = 1, ..., n.$$

(2.7) $$D_{x_1}^{\lambda_1 - \mu_1} ... D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} F_D^{(n)} [a, \mu_1, ..., \mu_n ; c; z_1 x_1, ..., z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} F_D^{(n)} [a, \lambda_1, ... \lambda_n; c; z_1 x_1, ..., z_n x_n],$$

$$max \left\{ |z_1 x_1|, ..., |z_n x_n| \right\} < 1, \ \mathrm{Re}(\lambda_i) > 0, \quad i = 1, ..., n.$$

(2.8) $$D_{x_1}^{\lambda_1 - \mu_1} ... D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} {}_{(1)}^{(k)}E_D^{(n)} [a, \lambda_1, ..., \lambda_n; c, c'; z_1 x_1, ..., z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} {}_{(i)}^{(k)}E_D^{(n)} [a, \lambda_1, ... \lambda_n; c, c'; z_1 x_1, ..., z_n x_n],$$

$$Re\ (\lambda_i) > 0, \ |z_i x_i| < r_i, \quad i = 1, ..., n; \quad r_1 = ... = r_k,$$

$$r_{k+1} = ... = r_n, r_k + r_n = 1.$$

(2.9) $$D_{x_1}^{\lambda_1 - \mu_1} ... D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} {}_{(2)}^{(k)}E_D^{(n)} [a, a', \mu_1, ..., \mu_n \right.$$

$$\left. c; z_1 x_1, ..., z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} {}_{(2)}^{(k)}E_D^{(n)} [a\ a', \lambda_1, ... \lambda_n; c; z_1 x_1, ..., z_n x_n],$$

$$Re\ (\lambda_i) > 0, \ |z_i x_i| < r_i, \quad i = 1, ..., n; \quad r_1 = ... = r_k,$$

$$r_{k+1} = ... r_n, r_k \,.\, r_n = r_k + r_n.$$

(2.10) $$D_{x_1}^{\lambda_1 - \mu_1} ... D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} {}_{(1)}^{(k)}E_C^{(n)} [a, a', \lambda_1, ..., \lambda_n; \right.$$

$$\left. ; z_1 x_1, ..., z_n x_n] \right\}$$

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$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \, {}^{(k)}_{(1)}E_C^{(n)} [a\, a', b\, \mu_1, ..., \mu_n; z_1 x_1, ..., z_n x_n],$$

$$Re\,(\lambda_i) > 0, \quad |z_i x_i| < r_i, \quad i = 1, ..., n; \quad (\sqrt{r_1} + ... + \sqrt{r_k})^2 + (\sqrt{r_{k+1}} + ... + \sqrt{r_n})^2 = 1.$$

$$(2.11) \quad D_{x_1}^{\lambda_1 - \mu_1} ... D_{x_1}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} \, {}^{(k)}F_{AC}^{(n)} [a, b, b_{k+1}, ..., b_n; \lambda_1, ..., \lambda_n; z_1 x_1, ..., z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \, {}^{(k)}F_{AC}^{(n)} [a\,, b, b_{k+1}, ..., b_n\,; \mu_1, ... \mu_n; z_1 x_1, ..., z_n x_n],$$

$$Re\,(\lambda_j) > 0, \; j = 1, ..., n.$$

$$(2.12) \quad D_{x_{k+1}}^{\lambda_{k+1} - \mu_{k+1}} ... D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=k+1}^{n} x_j^{\lambda_j - 1} \, {}^{(k)}F_{AC}^{(n)} [a, b, \mu_{k+1}, ..., \mu_n; c_1, ..., c_n; z_1, ..., z_k, z_{k+1} x_{k+1}, ..., z_n x_n] \right\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \, {}^{(k)}F_{AC}^{(n)} [a\,, b, \lambda_{k+1}, ..., \lambda_n\,; c_1, ..., c_n; z_1, ..., z_k, z_{k+1} x_{n+1}, ..., z_n x_n],$$

$$Re\,(\lambda_i) > 0, \; i = k+1, ..., n\,;$$

$$(|z_1|^{1/2} + ... + |z_k|^{1/2})^2 + |z_{k+1} x_{k+1}| + ... + |z_n x_n| < 1.$$

$$(2.13) \quad D_{x_1}^{\lambda_1 - \mu_1} ... D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} \, {}^{(k)}F_{AD}^{(n)} [a, \lambda_1, ..., \lambda_n; c\,; c_{k+1}, ..., c_n\,; z_1 x_1, ..., z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \, {}^{(k)}F_{AD}^{(n)} [a\,, \lambda_1, ..., \lambda_n\,; c; c_{k+1}, ..., c_n; z_1 x_1, ..., z_n x_n],$$

$$Re\,(\lambda_i) > 0, \; i = 1, ..., n.$$

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$$\max\{|z_1x_1|, ..., |z_kx_k|\} + |z_{k+1}x_{k+1} + ... + |z_nx_n| < 1$$

$$(2.14)\quad D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} ... D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=k+1}^{n} x_j^{\lambda_j-1}\ {}^{(k)}F_{AD}^{(n)}[a, b_1, ..., b_n; c; \lambda_{k+1}, ..., \lambda_n; z_1, ..., z_k, z_{k+1}x_{k+1}, ..., z_nx_n]\right\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1}\ {}^{(k)}F_{AD}^{(n)}[a, b_1, ..., b_n; c, \mu_{k+1}, ..., \mu_n; z_1, ..., z_k, z_{k+1}x_{k+1}, ..., z_nx_n],$$

$$\max\{|z_1|, ..., |z_k|\} + |z_{k+1}x_{k+1}| + ... + |z_nx_n| < 1,$$

$$Re(\lambda_i) > 0,\ i = 1, ..., n.$$

$$(2.15)\quad D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} ... D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=k+1}^{n} x_j^{\lambda_j-1}\ {}^{(k)}F_{BD}^{(n)}[a, \mu_{k+1}, ..., \mu_n, b_1, ..., b_n; c; z_1, ..., z_k, z_{k+1}x_{k+1}, ..., z_nx_n]\right\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1}\ {}^{(k)}F_{BD}^{(n)}[a, \lambda_{k+1}, ..., \lambda_n, b_1, ..., b_n; c; z_1, ..., z_k, z_{k+1}x_{k+1}, ..., z_nx_n],$$

$$\max\{|z_1|, ..., |z_k|, |z_{k+1}x_{k+1}|, ..., |z_nx_n|\} < 1,$$

$$Re(\lambda_i) > 0,\ i = k+1, ..., n.$$

$$(2.16)\quad D_{x_1}^{\lambda_1-\mu_1} ... D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\ {}^{(k)}F_{BD}^{(n)}[a, a_{k+1}, ..., a_n, \mu_1, ..., \mu_n; c; z_1x_1, ..., z_nx_n]\right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j-1}\ {}^{(k)}F_{BD}^{(n)}[a, a_{k+1}, ..., a_n, \lambda_1, ..., \lambda_n; c; z_1x_1, ..., z_nx_n],$$

$$max\ \{|z_1x_1|, ..., |z_nx_n|\} < 1, \qquad Re(\lambda_i) > 0,\ i = 1, ..., n.$$

$$(2.17)\quad D_{x_1}^{\lambda_1-\mu_1} ... D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n} x_j^{\lambda_j-1}\ {}^{(k)}F_{CD}^{(n)}[a, b, \mu_1, ..., \mu_k; c,\right.$$

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$$\lambda_{k+1}, \ldots, \lambda_n; z_1x_1, \ldots, z_nx_n]\Big\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \, {}^{(k)}F_{CD}^{(n)} [a, b, \lambda_1, \ldots, \lambda_k ; c, \mu_{k+1}, \ldots, \mu_n; z_1x_1, \ldots, z_nx_n],$$

$$Re\,(\lambda_i) > 0, \; i = 1, \ldots, n.$$

$$(2.18) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_n}^{\lambda_n - \mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j - 1} \; \Xi_1^{(n)} [\mu_1, \ldots, \mu_n, b_1, \ldots, b_{n-1}; \right.$$

$$c; z_1x_1, \ldots, z_nx_n]\Big\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \; \Xi_1^{(n)} [\lambda_1, \ldots, \lambda_n, b_1, \ldots, b_{n-1}; c; z_1x_1, \ldots, z_nx_n],$$

$$Re\,(\lambda_i) > 0, \; i = 1, \ldots, n.$$

$$(2.19) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_{n-1}}^{\lambda_{n-1} - \mu_{n-1}} \left\{ \prod_{j=1}^{n-1} x_j^{\lambda_j - 1} \; \Xi_1^{(n)} [a_1, \ldots, a_n, \mu_1, \ldots, \mu_{n-1}; \right.$$

$$c; z_1x_1, \ldots, z_{n-1}x_{n-1}, z_n]\Big\}$$

$$= \prod_{j=1}^{n-1} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \; \Xi_1^{(n)} [a_1, \ldots, a_n, \lambda_1, \ldots, \lambda_{n-1}; c;$$

$$z_1x_1, \ldots, z_{n-1}x_{n-1}, z_n],$$

$$Re\,(\lambda_i) > 0, \; i = 1, \ldots, n-1.$$

$$(2.20) \quad D_{x_1}^{\lambda_1 - \mu_1} \ldots D_{x_{n-1}}^{\lambda_{n-1} - \mu_{n-1}} \left\{ \prod_{j=1}^{n-1} x_j^{\lambda_j - 1} \; \Phi_3^{(n)} [\mu_1, \ldots, \mu_{n-1}; \right.$$

$$c; z_1x_1, \ldots z_{n-1}x_{n-1}, z_n]\Big\}$$

$$= \prod_{j=1}^{n-1} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \; \Phi_3^{(n)} [\lambda_1, \ldots, \lambda_{n-1}; c; \quad z_1x_1, \ldots, z_{n-1}x_{n-1}, z_n],$$

$$Re\,(\lambda_i) > 0, \; i = 1, \ldots, n-1.$$

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$$(2.21)\quad D_{x_1}^{\lambda_1-\mu_1}\ldots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n}x_j^{\lambda_j-1}\,\Psi_2^{(n)}[a,\lambda_1,\ldots,\lambda_n;z_1x_1,\ldots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\,\Psi_2^{(n)}[a,\mu_1,\ldots,\mu_n;z_1x_1,\ldots,z_nx_n].$$

$Re\,(\lambda_i)>0,\ i=1,\ldots,n.$

$$(2.22)\quad D_{x_1}^{\lambda_1-\mu_1}\ldots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n}x_j^{\lambda_j-1}\,\Phi_2^{(n)}[\mu_1,\ldots,\mu_n;c;z_1x_1,\ldots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\,\Phi_2^{(n)}[\lambda_1,\ldots,\lambda_n;c;z_1x_1,\ldots,z_nx_n],$$

$Re\,(\lambda_i)>0,\ i=1,\ldots,n.$

$$(2.23)\quad D_{x_1}^{\lambda_1-\mu_1}\ldots D_{x_{n-1}}^{\lambda_{n-1}-\mu_{n-1}}\left\{\prod_{j=1}^{n-1}x_j^{\lambda_j-1}\,\Phi_D^{(n)}[a,\mu_1,\ldots,\mu_{n-1},-;c;z_1x_1,\ldots,z_{n-1}x_{n-1},z_n]\right\}$$

$$=\prod_{j=1}^{n-1}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\,\Phi_D^{(n)}[a,\lambda_1,\ldots,\lambda_{n-1},-;c;z_1x_1,\ldots,z_{n-1}x_{n-1},z_n],$$

$Re\,(\lambda_i)>0,\ i=1,\ldots,n-1.$

$$(2.24)\quad D_{x_1}^{\lambda_1-\mu_1}\ldots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n}x_j^{\lambda_j-1}\,{}_{(1)}^{(k)}\Phi_{AC}^{(n)}[a,b;\lambda_1,\ldots,\lambda_n;z_1x_1,\ldots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\,{}_{(1)}^{(k)}\Phi_{AC}^{(n)}[a,b;\mu_1,\ldots,\mu_n;z_1x_1,\ldots,z_nx_n],$$

$Re\,(\lambda_i)>0,\ i=1,\ldots,n.$

$$(2.25)\quad D_{x_1}^{\lambda_1-\mu_1}\ldots D_{x_n}^{\lambda_n-\mu_n}\left\{\prod_{j=1}^{n}x_j^{\lambda_j-1}\,{}_{(2)}^{(k)}\Phi_{AC}^{(n)}[a,b_{k+1},\ldots,b_n;\lambda_1,\ldots,\lambda_n;z_1x_1,\ldots,z_nx_n]\right\}$$

$$=\prod_{j=1}^{n}\frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\,x_j^{\mu_j-1}\,{}_{(2)}^{(k)}\Phi_{AC}^{(n)}[a,b_{k+1},\ldots,b_n;\mu_1,\ldots,\mu_n;z_1x_1,\ldots,z_nx_n],$$

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$Re\,(\lambda_i) > 0,\ i = 1, \ldots, n.$

$$(2.26)\quad D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} \ldots D_{x_n}^{\lambda_n-\mu_n} \left\{ \prod_{j=k+1}^{n} x_j^{\lambda_j-1}\ {}_{(2)}^{(k)}\Phi_{AC}^{(n)}\,[a, \mu_{k+1}, \ldots, \mu_n;\ c_1, \ldots, c_n; z_1, \ldots, z_k, z_{k+1}x_{k+1}, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\, x_j^{\mu_j-1}\ {}_{(2)}^{(k)}\Phi_{AC}^{(n)}\,[a, \lambda_{k+1}, \ldots, \lambda_n\ ;\ c_1, \ldots, c_n\ ;\ z_1, \ldots z_k,\ z_{k+1}z_{k+1}, \ldots, z_n x_n],$$

$Re\,(\lambda_i) > 0,\ i = k+1, \ldots, n.$

$$(2.27)\quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j-1}\ {}_{(1)}^{(k)}\Phi_{AD}^{(n)}\,[a, \mu_1, \ldots, \mu_n; c; z_1x_1, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\, x_j^{\mu_j-1}\ {}_{(1)}^{(k)}\Phi_{AD}^{(n)}\,[a, \lambda_1, \ldots, \lambda_n\ ;\ c;\ z_1x_1, \ldots, z_n x_n],$$

$\mathrm{Re}\,(\lambda_i) > 0,\ i = 1, \ldots, n.$

$$(2.28)\quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_n-\mu_n} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j-1}\ {}_{(1)}^{(k)}\Phi_{BD}^{(n)}\,[a, \mu_1, \ldots, \mu_n; c; z_1x_1, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\, x_j^{\mu_j-1}\ {}_{(1)}^{(k)}\Phi_{BD}^{(n)}\,[a, \lambda_1, \ldots, \lambda_n\ ;\ c;\ z_1x_1, \ldots, z_n x_n],$$

$Re\,(\lambda_i) > 0,\ i = 1, \ldots, n.$

$$(2.29)\quad D_{x_{k+1}}^{\lambda_{k+1}-\mu_{k+1}} \ldots D_{x_n}^{\lambda_n-\mu_n} \left\{ \prod_{j=k+1}^{n} x_j^{\lambda_j-1}\ {}_{(2)}^{(k)}\Phi_{BD}^{(n)}\,[a, \mu_{k+1}, \ldots, \mu_n,\ , b_1, \ldots, b_n; c; z_1, \ldots, z_k, z_{k+1}x_{k+1}, \ldots, z_n x_n] \right\}$$

$$= \prod_{j=k+1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}\, x_j^{\mu_j-1}\ {}_{(2)}^{(k)}\Phi_{BD}^{(n)}\,[a, \lambda_{k+1}, \ldots, \lambda_n, b_1, \ldots, b_n;\ c; z_1, \ldots, z_k, z_{k+1}x_{k+1}, \ldots, z_n x_n],$$

$Re\,(\lambda_i) > 0,\ i = k+1, \ldots, n.$

$$(2.30)\quad D_{x_1}^{\lambda_1-\mu_1} \ldots D_{x_n}^{\lambda_{n-1}-\mu_{n-1}} \left\{ \prod_{j=1}^{n} x_j^{\lambda_j-1}\ {}_{(2)}^{(k)}\Phi_{BD}^{(n)}\,[a, a_{k+1}, \ldots, a_n,$$

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$$\mu_1, \ldots, \mu_n; c; z_1 x_1, \ldots, z_n x_n]\Big\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)} x_j^{\mu_j - 1} \; {}^{(k)}_{(2)}\Phi^{(n)}_{BD}\,[a, a_{k+1}, \ldots, a_n, \lambda_1, \ldots, \lambda_n;$$

$$c; z_1 x_1, \ldots, z_n x_n],$$

$$Re\,(\lambda_i) > 0, \quad i = 1, \ldots, n.$$

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CERTAIN GENERATING FUNCTIONS FOR THE KONHAUSER'S POLYNOMIALS $Z_n^{\alpha}(x; k)$

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ABSTRACT

In this paper, we derive some generating relations for one set $Z_n^{\alpha}(x; k)$ of the Konhauser biorthogonal polynomials. For these polynomials we have established some generating relations involving Kampe de Feriet's double hypergeometric function and Srivastava's general class of triple hypergeometric functions. These generating relations are deduced from certain well known results involving generalized hypergeometric polynomials.

1. INTRODUCTION

Konhauser [3] discussed two polynomial sets $Y_n^{\alpha}(x; k)$ and $Z_n^{\alpha}(x; k)$ which are biorthogonal with respect to the weight function $(x)^{\alpha}e^{-x}$ over the interval $(0, \infty)$, where $\alpha > -1$ and k is a positive integer $Y_n^{\alpha}(x; k)$ is a polynomial of degree n in x^k.

The biorthogonality relation is given by [3,p.303]

$$\int_0^{\infty} x^{\alpha}e^{-x}\, Y_n^{\alpha}(x; k)\, Z_m^{\alpha}(x;k)\, dx = \frac{\Gamma(km + \alpha + 1)}{m\,!}\, \delta_{nm}, \qquad \ldots (1.1)$$

where δ_{nm} is the Kronecker delta.

For $k = 1$, these polynomials reduce to the Laguerre polynomials $L_n^{(\alpha)}(x)$ and for $k = 2$, *these polynomials were studied by Spencer and Fano [9] in certain calculaions involving the penetration of gamma rays through matter and were subsequently discussed by Preiser [7].*

These polynomials were further investigated by Prabhakar [5,6], Srivastava [11] to [15], Patil and Thakare [4], Agarwal and Manocha [1] and, Srivastava and Singh [10].

One set $Z_n^{\alpha}(x; k)$ of the Konhauser's biorthogonal polynomials is given explicitly by [3].

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$$Z_n^{\alpha}(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + \alpha + 1)} \quad \ldots (1.2)$$

$\alpha > -1$ and k is a positive integer. An immediate consequence of (1.2) is the formula

$$Z_n^{\alpha}(x; k) = \frac{(\alpha + 1)_{kn}}{n!} \cdot {}_1F_k \left[\begin{matrix} -n; \\ (\Delta\, k; \alpha + 1); \end{matrix} \quad (x/k)^k \right] \quad \ldots (1.3)$$

where $\Delta(\lambda, \mu)$ represents the array of parametes $\frac{\mu}{\lambda} \frac{\mu+1}{\lambda}, \ldots, \frac{\mu+\lambda-1}{\lambda}$.

The Kampe de Feriet's double hypergeometric function (in the contracted notation of Burchnall and Chaundy [2, p. 112] is defined by

$$F^{(2)} \left[\begin{matrix} (a) : (b) : (c) : \\ (e) : (g) : (h) : \end{matrix} \quad x, y \right] = \sum_{m, n = 0}^{\infty} \frac{[(a)]_{m+n} [(b)]_m [(c)]_n \, x^m y^n}{[(e)]_{m+n} [(g)]_m [(h)]_n \, m! \, n!} \quad \ldots (1.4)$$

and a general class of triple hypergeometric functions, due to H.M.Srivastava, is defined by [11, p. 428]

$$F^{(3)} \left[\begin{matrix} (a) :: (b); (b') ; (b'') : (c) ; (c') ; (c'') ; \\ (e) :: (g) ; (g') ; (g'') : (h) ; (h') ; (h''); \end{matrix} \quad x, y, z \right]$$

$$= \sum_{l, m, n = 0}^{\infty} \frac{[(a)]_{l+m+n} [(b)]_{l+m} [(b')]_{m+n} [(b'')]_{n+l}}{[(e)]_{l+m+n} [(g)]_{l+m} [(g')]_{m+n} [(g''))]_{n+l}} \cdot$$

$$\frac{[(c)]_l [(c')]_m [(c'')]_n}{[(h)]_l [(h')]_m [(h'')]_n} \cdot \frac{x^l}{l!} \frac{y^m}{m!} \frac{z^n}{n!} \cdot \quad \ldots (1.5)$$

In the definitions of (1.4) and (1.5),as well as in what follows, (a) and $(b)_n$ abbreviate the sequence of A parameters $a_1, \ldots, a_A$ and the product $(a_1)_n, \ldots, (a_A)_n$. respectively.

Here, we deduce some generating relations for the polynomials $Z_n^{\alpha}(x; k)$ from some well known results for certain generalized hypergeometric polynomials (if, e.g.) Srivastava and Manocha (16) involving the function $F^{(2)}(x, y)$ and $F^{(3)}(x,y,z)$. All the generating relations, would reduce, when $k = 1$, to known results for Laguerre polynomials.

2. GENERATING RELATIONS

The generating relations that we deduce here from certain well known results are :

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$$\sum_{n=0}^{\infty} \frac{[(a)]_n}{[(e)]_n (\alpha+1)_{kn}} Z_n^{\alpha}(x;k) y^n$$

$$= F^{(2)}\left[\begin{matrix}(a): & ;-; \\ (e):(\alpha+1)/k, \ldots, (\alpha+k)/k; & -;\end{matrix} \quad y, -\{(x)^k/k \,.\, y\}\right] \ldots (2.1)$$

$$\sum_{n=0}^{\infty} \frac{[(a)]_n (1-\frac{\alpha+1}{k})_n \ldots (1-\frac{\alpha+k}{k})_n}{[(e)]_n (\alpha+1-kn)_{kn}} Z_n^{\alpha-kn}(x;k)\, y^n$$

$$= F^{(2)}\left[\begin{matrix}(a); -; 1-(\alpha+1)/k, \ldots, 1-(\alpha+k)/k; \\ (e): -; \text{————————};\end{matrix} \quad y, -\{(x/k)^k \,.\, y\}\right] \ldots (2.2)$$

$$= \sum_{n=0}^{\infty} \frac{[(a)]_m [(b'')]_n}{[(e)]_n [(g'')]_n (\alpha+1)_{kn}} {}_{A+C'}F_{E+H'}\left[\begin{matrix}(a)+n, (c') \\ (e)+n, (h');\end{matrix} \quad y\right] . Z_n^{\alpha}(x;k) z^n$$

$$= F^{(3)}\left[\begin{matrix}(a) :: -; (b''): \text{————}; (c); -; \\ (e) :: -; -; (g''):(\alpha+1)/k, \ldots, (\alpha+k)/k; (h'); -;\end{matrix} \quad y, z, -\{(x/k)^k \,.\, z\}\right] \ldots (2.3)$$

and

$$\sum_{n=0}^{\infty} \frac{[(a)]_n [(b'')]_n (1-\frac{\alpha+1}{k})_n \ldots (1-\frac{\alpha+k}{k})_n}{[(e)_n [(g'')]_n (\alpha+1-kn)_{kn}} .$$

$$F_{A+E'\ E+H'}\left[\begin{matrix}(a)+n, (e'); \\ (e)+n, (h');\end{matrix} \quad y\right] . Z_n^{\alpha-kn}(x;k) z^n$$

$$F^{(3)}\left[\begin{matrix}(a) :: -; -; (b''): -; (c'); 1-(\alpha+1)/k, \ldots 1-(\alpha+k)/k; \\ (e) :: -; -; (g'') -; (h'); \text{————————————};\end{matrix} \quad x, z, -\{(x/k)^k \,.\, z)\}\right] \ldots (2.4)$$

Derivations of (2.1) and (2.2) : From Srivastava and Manocha (16,p.194), and Rainville [18, p. 82], we have

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$$\sum_{n=0}^{\infty} \frac{[(a)]_n\,[(c)]_n}{[(e)]_n\,[(h)]_n}\, {}_{H+B+1}F_{C+G}\left[\begin{matrix} -n,\ 1-(h)-n,\ (b): \\ 1-(c)-n,\ (g); \end{matrix}\ x\right]\frac{y^n}{n!}$$

$$= F^{(2)}\left[\begin{matrix} (a) : (b)\ ;\ (c); \\ (e) : (g);\ (h); \end{matrix}\ y,\ [(-1)]^{c-h-1}\,.\,xy\right] \qquad \ldots (2.5)$$

Now the result (2.1) would follow at once if we interpret (2.5) in the light of (1.3).

In deducing (2.2), from the known result (2.5) we make use of

$$Z_n^{\alpha-kn}(x;k) = \frac{(\alpha+1-kn)_{kn}}{n!}\, {}_1F_k\left[\begin{matrix} -n; \\ \Delta(k;\alpha+1-kn); \end{matrix}\ (x/k)^k\right].$$

Derivations of (2.3) and (2.4) :

In case of the function $F^{(3)}(x, y, z)$, we recall from Srivastava and Manocha [16,p.157] that

$$\sum_{n=0}^{\infty} \frac{[(a)]_n\,[(b'')]_n\,[(c'')]_n}{[(e)]_m\,[(g'')]_n\,[(h'')]_n}\, {}_{A+C}F_{E+H'}\left[\begin{matrix} (a)+n,\ (c'); \\ (e)+n,\ (h'); \end{matrix}\ y\right]$$

$$.\ {}_{C+H''+1}F_{H+C''}\left[\begin{matrix} -n,\ (c),\ 1-(h'')-n; \\ (h),\ 1-(c'')-n; \end{matrix}\ x\right]\frac{z^n}{n!}$$

$$= F^{(3)}\left[\begin{matrix} (a) :: -;\ -;\ (b'') : (c);\ (c');\ (c''); \\ (e) :: -;\ -;\ (g'') : (h);\ (h');\ (h''); \end{matrix}\ y, z, xz\right] \qquad \ldots (2.6)$$

The specialised forms of (2.6) lead us to the desired generating relations (2.3) and (2.4).

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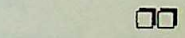

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INTEGRALS ASSOCIATED WITH GAUSS'S HYPERGEOMETRIC SERIES, MULTIVARIABLE *H*-FUNCTION AND A GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

A new class of integrals associated with hypergeometric series, the multivariable *H*-function and a general class of polynomials are evaluated. The results obtained are of general character and include the integrals given by Sharma and Rathie [9] and Arora and Rathie [1] etc.

1. INTRODUCTION

The object of this paper is to evaluate some integrals involving product of hypergeometric series,the multivariable *H*-function due to Srivastava and Panda [11] and a general class of polynomials due to Srivastava [10]. The integrals evaluated in this paper extend the results of Sharma and Rathie [9] and Arora and Rathie [1] etc.

The multivariable *H*-function introduced and studied by Srivastava and Panda [11] will be defined and represented in the following contracted form [12, pp.251-252, Eqns.] $(C.1)-(C.3)$]:

$$H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} (a_j:A_j^{(1)},\ldots,A_j^{(r)})_{1,p}:(c_j^{(1)},C_j^{(i)})_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ (b_j:B_j^{(1)},\ldots,B_j^{(r)})_{1,q}:(d_j^{(1)},D_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_i,\ldots,\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)\prod_{i=1}^{r}z_i^{\xi_i}\,d\xi_1\ldots d\xi_r \qquad \ldots(1.1)$$

where $w=\sqrt{-1}$,

$$\phi_i(\xi_i)=\frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-D_j^{(i)}\xi_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+C_j^{(i)}\xi_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+D_j^{(i)}\xi_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-C_j^{(i)}\xi_i)} \quad \forall\,(i=1,\ldots,r)$$

$$\ldots(1.2)$$

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$$\psi(\xi_1, \ldots, \xi_r) = \frac{\prod_{j=1}^{n} \Gamma\left(1 - a_j + \sum_{i=1}^{r} A_j^{(i)} \xi_i\right)}{\prod_{j=1}^{q} \Gamma\left(1 - b_j + \sum_{i=1}^{r} B_j^{(i)} \xi_i\right) \prod_{j=n+1}^{p} \Gamma\left(a_j - \sum_{i=1}^{r} A_j^{(i)} \xi_i\right)} \quad \ldots (1.3)$$

A detailed account of the multivariable H-function can be found in the monograh by Srivastava et al. [12].

Following Srivastava [10], the general class of polynomials is defined as

$$S_\beta^\alpha[x] = \sum_{u=0}^{[\beta/\alpha]} \frac{(-\beta)_{\alpha u}}{u!} A_{\beta, u}\, (x)^u; \quad \beta = 0, 1, 2, \ldots \quad \ldots (1.4)$$

where α is an arbitrary positive integer and the co-efficients $A_{\beta,u}$ $(\beta, u > 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{\beta,u}$, $S_\beta^\alpha[x]$ yeilds a number of known polynomials as its special cases [13,pp.158-161]. For convenience, the H-function of r complex variables defined by (1.1) will be denoted by the contracted notation $H[z_1, \ldots, z_r]$.

2.INTEGRALS

The following results are to be established here:

First Integral :

$$\int_0^1 t^{\rho-1}(1-t)^{\rho}\,[1 + at + b(1-t)]^{-2\rho-1}\, {}_2F_1\left[\nu, \delta; \tfrac{1}{2}(\nu + \delta + 2); \frac{t(1+a)}{1 + at + b(1-t)}\right]$$

$$\cdot S_\beta^\alpha\,[xR^k] H[z_1 R_1^{\lambda_1}, \ldots, z_r R_r^{\lambda_r}]\, dt$$

$$= \frac{2^{\nu+\delta-2\rho-1}\Gamma\left(\frac{\nu}{2} + \frac{\delta}{2} + 1\right)}{(\nu - \delta)(1+a)^{\rho}\,(1+b)^{\rho+1}\,\Gamma(\nu)\,\Gamma(\delta)} \sum_{u=0}^{[\beta/\alpha]} \frac{(-\beta)_{\alpha u}}{u!} A_{\beta, u}\, (x)^u$$

$$\left\{ \Gamma\left(\frac{\nu}{2} + \frac{1}{2}\right) \Gamma\left(\frac{\delta}{2}\right) H^{0, n+3; m_1, n_1; \ldots; m_r, n_r}_{p+3, q+3; p_1, q_1; \ldots; p_r, q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} I_1 \\ \\ I_2 \end{matrix} \right] \right.$$

$$\left. - \Gamma\left(\frac{\nu}{2}\right) \Gamma\left(\frac{\delta}{2} + \frac{1}{2}\right) H^{0, n+3; m_1, n_1; \ldots; m_r, n_r}_{p+3, q+3 : p_1, q_1; \ldots; p_r, q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} J_1 \\ \\ J_2 \end{matrix} \right] \right\}$$

where $\quad \ldots (2.1)$

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$$R = R_i = \frac{4(t + at)(1 + b)(1 - t)}{[1 + at + b(1 - t)]^2} \; ; \; \forall \; i \; \varepsilon \; \{1, 2, ..., r\}$$

... (2.2)

$$I_1 = (1 - \rho - ku; \lambda_1, ..., \lambda_r) \; ; (1 - \rho - ku + \frac{\nu}{2} + \frac{\delta}{2} \; ; \lambda_1, ..., \lambda_r);$$

$$\left(\frac{\nu}{2} - \frac{\delta}{2} - \rho - ku; \lambda_1, ..., \lambda_r\right); \left(a_j; A_j^{(1)}, ..., A_j^{(r)}\right)_{1, p} \; ;$$

$$(c_j^{(1)}, C_j^{(1)})_{1, p_1} \; ; ...; (c_j^{(r)}, C_j^{(r)})_{1, p_r} \qquad .... (2.3)$$

$$I_2 = (b_j \; ; B_j^{(1)}; \; ... \; ; B_j^{(r)})_{1, q}; (d_j^{(1)}, D_j^{(1)})_{1, q_1}; \; ... \; ; (d_j^{(r)}, D_j^{(r)})_{1, q_r};$$

$$(1 - \rho - ku + \frac{\nu}{2} - \frac{\delta}{2}; \lambda_1, ..., \lambda_r); (\frac{\nu}{2} - \rho - ku; \lambda_1, ..., \lambda_r);$$

$$(\frac{1}{2} + \frac{\delta}{2} - \rho - ku; \lambda_1, ..., \lambda_r). \qquad ... (2.4)$$

$$J_1 = (1 - \rho - ku; \lambda_1, ..., \lambda_r); (1 - \rho - ku + \frac{\nu}{2} + \frac{\delta}{2}; \lambda_1, ..., \lambda_r);$$

$$(\frac{\delta}{2} - \frac{\nu}{2} - \rho - ku; \lambda_1, ..., \lambda_r) \; ; (a_j \; ; A_j^{(1)} \; , ..., A_j^{(r)})_{1, p};$$

$$(c_j^{(1)}, C_j^{(1)})_{1, p_1} \; ; \; ... \; ; (c_j^{(r)}, C_j^{(r)})_{1, p_r} \qquad ... (2.5)$$

$$J_2 = (b_j \; ; B_j^{(1)}, ..., B_j^{(r)})_{1, q} \; ; (d_j^{(1)}, D_j^{(1)})_{1, q_1}; \; ...; (d_j^{(r)}, D_j^{(r)})_{1, q_r};$$

$$(1 - \rho - ku - \frac{\nu}{2} + \frac{\delta}{2} \; ; \lambda_1, ..., \lambda_r); (\frac{\delta}{2} - \rho - ku; \lambda_1, ..., \lambda_r);$$

$$(\frac{1}{2} + \frac{\nu}{2} - \rho - ku; \lambda_1, ..., \lambda_r). \qquad ... (2.6)$$

The (sufficient) conditions of validity of (2.1) are given below :

(*i*) The constants a and b are such that nons of the expressions $1 + a, 1 + b, 1 + at + (1 - t)$, where $0 \le t \le 1$, is zero. ... (2.7)

(*ii*) $\text{Re}(\rho) > 0, \text{Re}(2(\rho - \nu - \delta) > 0, \; k \ge 0.$ ... (2.8)

(*iii*) $\text{Re}(\rho + ku + \sum_{i=1}^{r} \lambda_i \, \xi_i) > 0, \; \lambda_j \ge 0$ ... (2.9)

and $\xi_i = \min_{i \le j \le m_i} [\text{Re}(d_j^{(i)}/D_j^{(i)})]$ where $u = 0, 1, ..., [\beta/\alpha]$

$\forall \; i \; \varepsilon \; \{1, 2, ..., r\} \; j = 1, 2, ..., m_r.$

(*iv*) $\Omega_i > 0, \; |\arg z_i| < \frac{1}{2} \pi \, \Omega_i \; ; \forall \; i \; \varepsilon \; \{1, 2, ..., r\}$ ... (2.10)

where

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$$\Omega_i = -\sum_{j=n+1}^{p} A_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)} - \sum_{j=1}^{q} B_j^{(i)}$$

$$+ \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} > 0; \forall\, i \in \{1, 2, ..., r\}$$

Second Integral :

$$\int_0^1 t^{\rho-1}(1-t)^{p-2}\,[1 + at + b(1-t)]^{-2\rho+1}\; {}_2F_1[\nu, \delta; \tfrac{1}{2}(\nu+\delta);$$

$$\frac{t(1+a)}{1+at+b(1-t)}]\; S_\beta^\alpha\,[xR^k]H[z_1R_1^{\lambda_1}, ..., z_rR_r^{\lambda_r}]\,dt$$

$$= \frac{2^{\nu+\delta-2\rho}\,\Gamma(\frac{\nu}{2}+\frac{\delta}{2})}{(1+a)^\rho(1+b)^{\rho-1}\,\Gamma(\nu)\,\Gamma(\delta)} \sum_{u=0}^{[\beta/\alpha]} \frac{(-\beta)_{\alpha u}}{u\,!} A_{\beta,u}(x)^u$$

$$\cdot\left\{\Gamma(\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\delta}{2})\,H^{0,\,n+3;\,m_1,n_1;\,...;\,m_r,n_r}_{p+3,\,q+3;\,p_1,q_1;\,...;\,p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} E_1 \\ \\ E_2 \end{matrix}\right]\right.$$

$$\left. + \Gamma(\frac{\nu}{2})\,\Gamma(\frac{\delta}{2}+\frac{1}{2})\,H^{0,\,n+3;\,m_1,n_1;\,...;\,m_r,n_r}_{p+3,\,q+3;\,p_1,q_1;\,...;\,p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} G_1 \\ \\ G_2 \end{matrix}\right]\right\} \qquad \text{... (2.11)}$$

where

$R, R_1, ..., R_r$ are defined by (2.2).

$$E_1 = (2-\rho-ku\,;\lambda_1, ..., \lambda_r);\ (2-\rho-ku+\frac{\nu}{2}+\frac{\delta}{2}\,;\lambda_1, ..., \lambda_r);$$

$$(1+\frac{\nu}{2}-\frac{\delta}{2}-\rho-ku\,;\lambda_1, ..., \lambda_r);\ (a_j\,; A_j^{(1)}, ..., A_j^{(r)})_{1,p}\,;$$

$$(c_j^{(1)}, C_j^{(1)})_{1,p_1}\,; ...\,; (c_j^{(r)}, C_j^{(r)})_{1,p_r} \qquad \text{... (2.12)}$$

$$E_2 = (b_j\,; B_j^{(1)}, ..., B_j^{(r)})_{1,q};\ (d_j^{(1)})_{1,q_1}\,; ...; (d_j^{(r)}, D_j^{(r)})_{1,q_r};$$

$$(2-\rho-ku+\frac{\nu}{2}-\frac{\delta}{2};\lambda_1, ..., \lambda_r);\ (1+\frac{\nu}{2}-\rho-ku;\lambda_1, ..., \lambda_r);$$

$$(\frac{3}{2}+\frac{\delta}{2}-\rho-ku;\lambda_1, ..., \lambda_r) \qquad \text{... (2.13)}$$

$$G_1 = (2-\rho-ku\,;\lambda_1, ..., \lambda_r);\ (2-\rho-ku+\frac{\nu}{2}+\frac{\delta}{2}\,;\lambda_1, ..., \lambda_r);$$

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$$(1+\frac{\delta}{2}-\frac{\nu}{2}-\rho-ku\,;\lambda_1,\ldots,\lambda_r);(a_j\,;A_j^{(1)},\ldots,A_j^{(r)})_{1,p}\,;$$

$$(c_j^{(1)},C_j^{(1)})_{1,p_1}\,;\ldots\,;(c_j^{(r)},C_j^{(r)})_{1,p_r} \qquad \ldots (2.14)$$

$$G_2=(b_j\,;B_j^{(1)},\ldots,B_j^{(r)})_{1,q};(d_j^{(1)},D_j^{(1)})_{1,q_1}\,;\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r};$$

$$(2-\rho-ku-\frac{\nu}{2}+\frac{\delta}{2};\lambda_1,\ldots,\lambda_r);(1+\frac{\delta}{2}-\rho-ku;\lambda_1,\ldots,\lambda_r);$$

$$(\frac{3}{2}+\frac{\nu}{2}-\rho-ku;\lambda_1,\ldots,\lambda_r) \qquad \ldots (2.15)$$

The (sufficient) conditions of validity of (2.11) are given below :

(*i*) The conditions (2.7) and (2.10) hold.

(*ii*) $\mathrm{Re}(\rho)>1$, $\mathrm{Re}(2\rho-\nu-\delta)>2$; $k\geq 0$.

(*iii*) $\mathrm{Re}(\rho+ku+\sum_{i=1}^{r}\lambda_i\,\xi_i)>1$; $\lambda_j\geq 0\ \forall\ j$, $\qquad \ldots (2.16)$

where

$$\xi_i=\min_{1\leq j\leq m_i}[\mathrm{Re}(d_j^{(i)}/D_j^{(i)})]\ \ \forall\ i\in\{1,2,\ldots,r\} \qquad \ldots (2.17)$$

Third Integral :

(2.11)

$$\int_0^{\pi/2} e^{w(2\rho+1)\theta}\,(\sin\theta)^{\rho}\,(\cos\theta)^{\rho-1}\,{}_2F_1[\nu,\delta;\tfrac{1}{2}(\nu+\delta+2)\,e^{w\theta}\cos\theta]$$

$$.\,S_\beta^\alpha[xT^k]H[z_1T_1^{\lambda_1},\ldots,z_rT_r^{\lambda_r}]d\theta$$

$$=\frac{e^{(w(\rho+1)/2)\pi}\,\Gamma(\frac{\nu}{2}+\frac{\delta}{2}+1)}{2^{2\rho-\nu-\delta+1}\,\Gamma(\nu)\,\Gamma(\delta)\,\Gamma(\nu-\delta)}\sum_{u=0}^{[\beta/\alpha]}\frac{(-\beta)_{\alpha u}}{(u)\,!}A_{\beta,u}(x)^u$$

(2.12)

$$\left\{\Gamma(\nu+1)/2)\,\Gamma(-\delta/2)\,H^{0,n+3;m_1,n_1;\ldots;m_r,n_r}_{p+3,q+3;p_1,q_1,\ldots;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}I_1\\ \\ I_2\end{matrix}\right]\right.$$

$$\left.-\Gamma(\frac{\nu}{2})\,\Gamma(\frac{\delta}{2}+\frac{1}{2})H^{0,n+3;m_1,n_1;\ldots;m_r,n_r}_{p+3,q+3;p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}J_1\\ \\ J_2\end{matrix}\right]\right\} \qquad \ldots (2.18)$$

(2.13)

where I_1, I_2, J_1 and J_2 are defined by (2.3), (2.4), (2.5), and (2.6) respectively and $w=\sqrt{-1}$.

Also

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$$T = T_i = \frac{4e^{2w\theta} \sin\theta \cos\theta}{e^{w\pi/2}} ; \forall\, i \in \{1, 2, ..., r\} \qquad ... (2.19)$$

The (sufficient) conditions for the validity of the integral (2.18) are given below :

(i) $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(2\rho - \nu - \delta) > 0$, $k \geq 0$;

(ii) $\mathrm{Re}(\rho + ku + \sum_{i=1}^{r} \lambda_i \xi_i) > 0$; $\lambda_i \geq 0\ \forall\, j$;

where ξ_i is defined by (2.17).

Fourth Integral :

$$\int_0^{\pi/2} e^{w(2\rho-1)\theta} (\sin\theta)^{\rho-2} (\cos\theta)^{\rho-1}\, {}_2F_1[\nu, \delta; (\frac{\nu+\delta}{2}) ; e^{w\theta} \cos\theta]$$

$$. S_\beta^\alpha[xT^k] H[z_1 T_1^{\lambda_1}, ..., z_r T_r^{\lambda_r}] d\theta$$

$$= \frac{e^{w\pi(\rho-1)/2}\, \Gamma(\frac{\nu}{2} + \frac{\delta}{2})}{2^{2\rho-\nu-\delta}\, \Gamma(\nu)\, \Gamma(\delta)} \sum_{u=0}^{[\beta/\alpha]} \frac{(-\beta)_{\alpha u}}{(u)\,!} A_{\beta,u}(x)^u$$

$$. \left\{ \Gamma(\frac{\nu}{2} + \frac{1}{2})\, \Gamma(\frac{\delta}{2})\, H^{0, n+3; m_1, n_1; ..., m_r, n_r}_{p+3, q+3; p_1, q_1; ...; p_r, q_r} \left[\begin{matrix} z_1 & E_1 \\ \vdots & \\ z_r & E_2 \end{matrix} \right] \right.$$

$$\left. + \Gamma(\frac{\nu}{2})\, \Gamma(\frac{\delta}{2} + \frac{1}{2}) H^{0, n+3; m_1, n_1; ...; m_r, n_r}_{p+3, q+3; p_1, q_1; ...; p_r, q_r} \left[\begin{matrix} z_1 & G_1 \\ \vdots & \\ z_r & G_2 \end{matrix} \right] \right\} \qquad ... (2.20)$$

where $T, T_1, ..., T_r$ are defined by (2.19) and E_1, E_2, G_1 and G_2 are defined by (2.12), (2.13), (2.14) and (2.15) respectively.

The (sufficient) conditions for the validity of the integral (2.20) are given below :

(i) $\mathrm{Re}(\rho) > 1$, $\mathrm{Re}(2\rho - \nu - \delta) > 2$, $k \geq 0$.

(ii) $\mathrm{Re}(\rho + ku - 2 + \sum_{i=1}^{r} \lambda_i \xi_i) > 0$; $\lambda_i \geq 0\ \forall\, j$ .... (2.21)

where ξ_i is defined by (2.17).

(iii) The (2.16) also hold.

Fifth Integral :

$$\int_0^{\pi/2} e^{w(2\rho+1)\theta} (\sin\theta)^{\rho-1} (\cos\theta)^{\rho}\, {}_2F_1[\nu, \delta; \tfrac{1}{2}(\nu + \delta + 2);$$

$$e^{w(\theta - \pi/2)} \sin\theta]\, S_\beta^\alpha\, [XT^k]\, H\, [z_1 T_1^{\lambda_1}, ..., z_r T_r^{\lambda_r}]\, d\theta$$

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$$= \frac{e^{\omega\pi\rho/2}\,\Gamma(\nu/2+\delta/2+1)}{2^{2\rho-\nu-\delta+1}\,(\nu-\delta)\,\Gamma(\nu)\,\Gamma(\delta)} \sum_{u=0}^{[\beta/\alpha]} \frac{(-\beta)_{\alpha u}}{u!} A_{\beta,u}(x)^u$$

$$\left\{ \Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\delta}{2}\right) H^{0,\,n+3;\,m_1,n_1;\ldots;\,m_r,n_r}_{p+3,\,q+3;\,p_1,q_1;\ldots;\,p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} I_1 \\ \\ I_2 \end{matrix}\right] \right.$$

$$\left. - \Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\delta}{2}+\frac{1}{2}\right) H^{0,\,n+3;\,m_1,n_1;\ldots;\,m_r,n_r}_{p+3,\,q+3;\,p_1,q_1;\ldots;\,p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} J_1 \\ \\ J_2 \end{matrix}\right] \right\} \qquad \ldots (2.22)$$

where $T, T_1, T_2, \ldots, T_r$ are defined by (2.19).

and I_1, I_2, J_1 and J_2 are defined by (2.3), (2.4), (2.5) and (2.6) respectively.

The (sufficient) conditions for the validity of the integral (2.22) are given below :

(*i*) $\mathrm{Re}(\rho) > 0;\ \mathrm{Re}(2\rho - \nu - \delta) > 0;\ k \geq 0.$

(*ii*) The condition (2.16) holds.

Sixth Integral :

$$\int_0^{\pi/2} e^{\omega(2\rho-1)\theta}\,(\sin\theta)^{\rho-1}\,(\cos\theta)^{\rho-2}\;{}_2F_1[\nu,\delta;\tfrac{1}{2}(\nu+\delta);$$

$$e^{\omega(\theta-\pi/2)}\sin\theta]\;S^{\alpha}_{\beta}\,[XT^k]\;H\,[z_1T_1^{\lambda_1},\ldots,z_rT_r^{\lambda_r}]d\theta$$

$$= \frac{e^{\omega\pi\rho/2}\,\Gamma\left(\frac{\nu}{2}+\frac{\delta}{2}\right)}{2^{2\rho-\nu-\delta}\,\Gamma(\nu)\,\Gamma(\delta)} \sum_{u=0}^{[\beta/\alpha]} \frac{(-\beta)_{\alpha u}}{u!}\,A_{\beta,u}(x)^u$$

$$\cdot\left\{\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\delta}{2}\right) H^{0,\,n+3;\,m_1,n_1;\ldots;\,m_r,n_r}_{p+3,\,q+3,\,p_1,q_1;\ldots;\,p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} E_1 \\ \\ E_2 \end{matrix}\right]\right.$$

$$\left. + \left(\frac{\nu}{2}\right)\Gamma\left(\frac{\delta}{2}+\frac{1}{2}\right) H^{0,\,n+3;\,m_1,n_1;\ldots;\,m_r,n_r}_{p+3,\,q+3,\,p_1,q_1;\ldots;\,p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} G_1 \\ \\ G_2 \end{matrix}\right]\right\} \qquad \ldots (2.23)$$

where T, T_i are defined by (2.19) and E_1, E_2, G_1 and G_2 are defined by (2.12),(2.13),(2.14) and (2.15) respectively.

The (sufficient) conditions of validity of the integral (2.23) are given below :

(*i*) $\mathrm{Re}(\rho) > 1,\ \mathrm{Re}(2\rho - \nu - \delta) > 2;\ k \geq 0.$

(*ii*) The conditions (2.16) and (2.21) also hold.

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Seventh Integral :

$$\int_0^1 t^{\sigma-1}(1-t)^{\sigma-\mu-1}\,{}_2F_1[\nu,-\nu;\mu;t]\,S_\beta^\alpha[xt^k(1-t)^k]$$

$$.H[z_1R_1;\ldots;z_rR_r]dt$$

$$=\frac{2^{-2\nu-1}\Gamma(\mu)}{\Gamma(\mu-\nu)}\sum_{u=0}^{[\beta/\alpha]}\frac{(-\beta)_{\alpha u}}{u!}A_{\beta,u}(x)^u\left\{\frac{\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})}\right.$$

$$.H^{0,n+3;m_1,n_1;\ldots;m_r,n_r}_{p+3,q+2;p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}L_1\\ \\ L_2\end{matrix}\right]-\frac{\Gamma(\frac{\mu}{2}-\frac{\nu}{2})}{\Gamma(\frac{\mu}{2}+\frac{\nu}{2})}$$

$$\left..H^{0,n+3;m_1,n_1;\ldots;m_r,n_r}_{p+3,q+2;p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}L_1^*\\ \\ L_2^*\end{matrix}\right]\right\} \quad \ldots (2.24)$$

where $R_i=[t(1-t)^{\lambda_i}\ ;\ \forall\, i\,\varepsilon\,\{1,2,\ldots,r\}$ ... (2.25)

$$L_1=(1-\sigma-ku;\lambda_1,\ldots,\lambda_r)\ ;(1-\sigma+\mu-ku;\lambda_1,\ldots,\lambda_r);$$

$$(1-\sigma+\frac{\mu}{2}+\frac{\nu}{2}-ku;\lambda_1,\ldots,\lambda_r);(a_j;A_j^{(1)}),\ldots,A_j^{(r)})_{1,p}$$

$$.(c_j^{(1)},C_j^{(1)})_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \quad \ldots (2.26)$$

$$L_2=(b_j;B_j^{(1)},\ldots,B_j^{(r)})_{1,q};(d_j^{(1)},D_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r};$$

$$(1-2\sigma+\mu+\nu-2ku;2\lambda_1,\ldots,2\lambda_r);(1-\sigma+\frac{\mu}{2}-\frac{\nu}{2}-ku;\lambda_1,\ldots,\lambda_r)$$

... (2.27)

$$L_1^*=(1-\sigma-ku;\lambda_1,\ldots,\lambda_r)\ ;(1-\sigma+\mu-ku;\lambda_1,\ldots,\lambda_r);$$

$$(\tfrac{1}{2}-\sigma-ku+\frac{\mu}{2}+\frac{\nu}{2};\lambda_1,\ldots,\lambda_r);(a_j;A_j^{(1)},\ldots,A_j^{(r)})_{1,p};$$

$$.(c_j^{(1)},C_j^{(1)})_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \quad \ldots (2.28)$$

$$L_2^*=(b_j;B_j^{(1)},\ldots,B_j^{(r)})_{1,q};(d_j^{(1)},D_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r};$$

$$(1-2\sigma+\mu+\nu-2ku;2\lambda_1,\ldots,2\lambda_r);(\tfrac{1}{2}-\sigma+\frac{\mu}{2}-\frac{\nu}{2}-ku;\lambda_1,\ldots,\lambda_r)$$

... (2.29)

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The (sufficient) conditions of validity of (2.24) are given below:

(*i*) $\mathrm{Re}(\sigma) > 0, \mathrm{Re}(\mu) > 0, \mathrm{Re}(\sigma - \mu) > 0, k \geq 0.$ ... (2.30)

(*ii*) $\mathrm{Re}(\sigma + ku + \sum_{i=1}^{r} \lambda_i \xi_i) > 0, \mathrm{Re}(\sigma - \mu + ku + \sum_{i=1}^{r} \lambda_i \xi_i) > 1;$

... (2.31)

$\lambda_j \geq 0 \; \forall \; j; u = 0, 1, ..., (\beta/\alpha)$ and $j = 1, 2, ..., m_r$.

and

$$\xi_i = \min_{1 \leq j \leq m_i} [\mathrm{Re}(\frac{d_j^{(i)}}{D_j^{(i)}})], \; \forall \; i \in 1, 2, ..., r) \qquad \text{... (2.32)}$$

(*iii*) The condition (2.10) also holds.

Eighth Integral :

$$\int_0^1 t^{\sigma-1} (1-t)^{\sigma-\mu-1} \, {}_2F_1[\nu, -1-\nu; \mu; t] \, S_\beta^\alpha [xt^k (1-t)^k]$$

$$. H[z_1 R_1; ...; z_r R_r] dt$$

$$= \frac{2^{-2\nu-2} \Gamma(\mu)}{\Gamma(\mu-\nu)} \sum_{u=0}^{[\beta/\alpha]} \frac{(-\beta)_{\alpha u}}{u!} A_{\beta, u} (x)^u \left\{ \frac{2 \Gamma(\frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2})}{\Gamma(\frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2})} \right.$$

$$. H^{0, n+4; m_1, n_1; ...; m_r, n_r}_{p+4, q+3; p_1, q_1; ...; p_r, q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} L_3 \\ \\ L_4 \end{matrix} \right] - \frac{\mu \, \Gamma(\frac{\mu}{2} - \frac{\nu}{2})}{\Gamma(\frac{\mu}{2} + \frac{\nu}{2} + 1)}$$

$$\left. . H^{0, n+3; m_1, n_1; ...; m_r, n_r}_{p+3, q+2; p_1, q_1; ...; p_r, q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} L_1^* \\ \\ L_2^* \end{matrix} \right] \right\} \qquad \text{... (2.33)}$$

where $R_1, ..., R_r$ are defined by (2.25) and L_1^* and L_2^* are defined by (2.28) and (2.29) respectively.

$L_3 = (1 - \sigma - ku; \lambda_1, .., \lambda_r); (1 - \sigma + \mu - ku; \lambda_1, ..., \lambda_r);$

$$(\frac{\mu}{2} - \sigma - ku; \lambda_1, .., \lambda_r); (1 - \sigma + \frac{\mu}{2} + \frac{\nu}{2} - ku; \lambda_1, ..., \lambda_r);$$

$$(a_j; A_j^{(1)}, ..., A_j^{(r)})_{1, p}; (c_j^{(1)}, C_j^{(1)})_{1, p_1}; ..., (c_j^{(r)}, C_j^{(r)})_{1, p_r} \qquad \text{... (2.34)}$$

$L_4 = (b_j; B_j^{(1)}; ...; B_j^{(r)})_{1, q}; (d_j^{(1)}, D_j^{(1)})_{1, q}; ...; (d_j^{(r)}, D_j^{(r)})_{1, q};$

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$$(1-2\sigma+\mu+\nu-2ku;\ 2\lambda_1,\ ...,\ 2\lambda_r);\ (1-\sigma+\frac{\mu}{2}-ku;\ \lambda_1,\ ...,\ \lambda_r)$$

$$.(\frac{\mu}{2}-\frac{\nu}{2}-\sigma-ku;\ \lambda_1,\ ...,\ \lambda_r). \qquad ... (2.35)$$

The (sufficient) conditions of validity of (2.33) are given below :

(*i*) The conditions (2.10), (2.30) and (3.31) also hold.

Ninth Integral :

$$\int_0^1 t^{\sigma-1}(1-t)^{\sigma-\mu-1}\ {}_2F_1[\nu,\ 1-\nu;\ \mu;\ t]\ S_\beta^\alpha\ [x\,t^k(1-t)^k]$$

$$.H[z_1R_1;\ ...;\ z_rR_r]dt$$

$$=\frac{2^{-2\nu}\Gamma(\mu)}{\Gamma(\mu-\nu)}\sum_{u=0}^{[\beta/\alpha]}\frac{(-\beta)_{\alpha\mu}}{u\,!}A_{\beta,\,u}\,(x)^u$$

$$\left\{\frac{\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\frac{\mu}{2}+\frac{\nu}{2}-\frac{1}{2})}H^{0,\,n+3\,;\,m_1,\,n_1;\,...;\,m_r,\,n_r}_{p+3,\,q+2;\,p_1,\,q_1;\,...;\,p_r,\,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}L_5\\ \\ L_6\end{matrix}\right]\right.$$

$$\left.+\frac{\Gamma(\frac{\mu}{2}-\frac{\nu}{2})}{\Gamma(\frac{\mu}{2}+\frac{\nu}{2})}H^{0,\,n+3;\,m_1,\,n_1;\,...;\,m_r,\,n_r}_{p+3,\,q+2;\,p_1,\,q_1;\,...;\,p_r,\,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}L_5^*\\ \\ L_6^*\end{matrix}\right]\right\} \qquad ... (2.36)$$

where $R_1, R_2, ..., R_r$ defined by (2.25).

$$L_5=(2-\sigma-ku;\ \lambda_1,\ ...,\ \lambda_r);\ (1-\sigma+\mu-ku;\ \lambda_1,\ ...,\ \lambda_r);$$

$$(1-\sigma+\frac{\mu}{2}+\frac{\nu}{2}-ku;\ \lambda_1,\ ...,\ \lambda_r);\ (a_j;\ A_j^{(1)},\ ...A_j^{(r)})_{1,\,p};$$

$$(c_j^{(1)},C_j^{(1)})_{1,\,p_1};\ ...;\ (c_j^{(r)},\ C_j^{(r)})_{1,\,p_r} \qquad ... (2.37)$$

$$L_6=(b_j;\ B_j^{(1)},\ ...,\ B_j^{(r)})_{1,\,q};\ (d_j^{(1)},D_j^{(1)})_{1,\,q_1};\ ...;\ (d_j^{(r)},\ D_j^{(r)})_{1,q_r};$$

$$(1-2\sigma+\mu+\nu-2ku;\ 2\lambda_1,\ ...,\ 2\lambda_r);\ (1-\sigma+\frac{\mu}{2}-\frac{\nu}{2}-ku;\ \lambda_1,\ ...,\ \lambda_r)$$

$$... (2.38)$$

$$L_5^*=(2-\sigma-ku;\ \lambda_1,\ ...,\ \lambda_r);\ (1-\sigma+\mu-ku;\ \lambda_1,\ ...,\ \lambda_r);$$

$$(\tfrac{1}{2}-\sigma+\frac{\mu}{2}+\frac{\nu}{2}+\frac{\nu}{2}-ku;\ \lambda_1,\ ...,\ \lambda_r);\ (a_j;\ A_j^{(1)},\ ...,\ A_j^{(r)})_{1,\,p};$$

$$(c_j^{(1)},C_j^{(1)})_{1,\,p_1},\ ...,\ (c_j^{(r)},C_j^{(r)})_{1,\,p_r} \qquad ... (2.39)$$

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$$L_6^* = (b_j; B_j^{(1)}, ..., B_j^{(r)})_{1,q}; (d_j^{(1)}, D_j^{(1)})_{1,q_1}; ...; (d_j^{(r)}, D_j^{(r)})_{1,q_r};$$

$$(1 - 2\sigma + \mu + \nu - 2ku; 2\lambda_1, ..., 2\lambda_r); (\tfrac{3}{2} - \sigma + \frac{\mu}{2} - \frac{\nu}{2} - ku; \lambda_1, ..., \lambda_r)$$

... (2.40)

The (sufficient) conditions of validity of (2.36) are given below:

(i) The conditions (2.10), (2.30) and (2.31) hold .

Proof of (2.1) : To evaluate the integral (2.1), we first express the multivariable H-function in terms of multiple Mellin-Barnes type contour integral (1.1), then on using (1.4) and interchanging the order of ξ_i-integrals and t-integral which is permissible under the conditions stated on account of the absolute (and uniform) convergence of the integrals and finally evaluating the t-integral with the help of known result [(9,pp.26,Eq. (2.1)], the result readily follows.

The remaining integrals (2.11), (2.18), (2.20), (2.22) and (2.23) can be proved in the same way by employing the integrals [9, pp.26-28, Eqs. (2.2) to (2.6)], respectively.The following integrals (2.24), (2.33) and (2.36), can be established by using the results [1,pp.84-86, Eqs. (5.1) to (5.3)] respectively.

3. SPECIAL CASES

(1) If we have $n = p = q = 0$, the multivariable H-function in(2.1) breaks up into products of r, H-functions and following integral is obtained :

$$\int_0^1 t^{\rho-1}(1-t)^{\rho}\,[1 + at + b(1-t)]^{-2\rho-1}\,{}_2F_1\left[\nu, \delta; \tfrac{1}{2}(\nu+\delta+2); \frac{t(1+a)}{1+at+b(1-t)}\right]$$

$$. S_{\beta}^{\alpha}[xR^k] \prod_{i=1}^{r} H_{p_i, q_i}^{m_i, n_i}\left[z_i R_i^{\lambda_i} \middle| \begin{matrix}(c_j^{(i)}, C_j^{(i)})_{1,p_i} \\ (d_j^{(i)}, D_j^{(i)})_{1,q_i}\end{matrix}\right] dt$$

$$= \frac{2^{\nu+\delta-2\rho-1}\,\Gamma(\frac{\nu}{2}+\frac{\delta}{2}+1)}{(\nu-\delta)(1+a)^{\rho}\,(1+b)^{\rho+1}\,\Gamma(\nu)\,\Gamma(\delta)} \sum_{u=0}^{[\beta/\alpha]} \frac{(-\beta)_{\alpha u}}{u!} A_{\beta,u}\,(x)^u$$

$$\left\{\Gamma(\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\delta}{2})\,H_{3,3;p_1,q_1;...;p_r,q_r}^{0,3;m_1,n_1;...;m_r,n_r}\left[\begin{matrix}z_1 \\ \vdots \\ z_r\end{matrix}\middle|\begin{matrix}I'_1 \\ \\ I'_2\end{matrix}\right]\right.$$

$$\left. - \Gamma(\frac{\nu}{2})\,\Gamma(\frac{\delta}{2}+\frac{1}{2})\,H_{3,3;p_1,q_1;...;p_r,q_r}^{0,3;m_1,n_1;...;m_r,n_r}\left[\begin{matrix}z_1 \\ \vdots \\ z_r\end{matrix}\middle|\begin{matrix}J'_1 \\ \\ J'_2\end{matrix}\right]\right\} \quad \text{... (3.1)}$$

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where $R = R_1 = R_2 = ... = R_i$ are defined by (2.2)

$$I'_1 = (1-\rho-ku; \lambda_1, ..., \lambda_r); (1-\rho-ku+\frac{\nu}{2}+\frac{\delta}{2}; \lambda_1, ..., \lambda_r)$$

$$. (\frac{\nu}{2}-\frac{\delta}{2}-\rho-ku; \lambda_1, ..., \lambda_r); (c_j^{(1)}, C_j^{(1)})_{1,p_1}; ...; (c_j^{(r)}, C_j^{(r)})_{1,p_r} \quad ... (3.2)$$

$$I'_2 = (d_j^{(1)}, D_j^{(1)})_{1,q_1}; ...; (d_j^{(r)}, D_j^{(r)})_{1,q_r}; (1-\rho-ku+\frac{\nu}{2}-\frac{\delta}{2}; \lambda_1, ..., \lambda_r);$$

$$(\frac{\nu}{2}-\rho-ku; \lambda_1, ..., \lambda_r); (\frac{1}{2}+\frac{\delta}{2}-\rho-ku; \lambda_1, ..., \lambda_r) \quad ... (3.3)$$

$$J'_1 = (1-\rho-ku; \lambda_1, ..., \lambda_r); (1-\rho-ku+\frac{\nu}{2}+\frac{\delta}{2}; \lambda_1, ..., \lambda_r)$$

$$(\frac{\delta}{2}-\frac{\nu}{2}-\rho-ku; \lambda_1, ..., \lambda_r); (c_j^{(1)}, C_j^{(1)})_{1,p_1}; ...; (c_j^{(r)}, C_j^{(r)})_{1,p_r} \quad ... (3.4)$$

$$J'_2 = (d_j^{(1)}, D_j^{(1)})_{1,q_1}; ...; (d_j^{(r)}, D_j^{(r)})_{1,q_r}; (1-\rho-ku-\frac{\nu}{2}+\frac{\delta}{2};$$

$$\lambda_1, ..., \lambda_r); (\frac{\delta}{2}-\rho-ku; \lambda_1, ..., \lambda_r); (\frac{1}{2}+\frac{\nu}{2}-\rho-ku; \lambda_1, ..., \lambda_r) ... (3.5)$$

The (sufficient) conditions of validity of (3.1) are given below :

(*i*) The conditions (2.7),(2.8) and (2.9) hold.

(*ii*) $\Omega_i^* > 0$, $|\arg z_i| < \frac{1}{2}\pi\,\Omega_i^*$; $\forall\, i \in \{1, 2, ..., r\}$.

where

$$\Omega_i^* = \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{i=1}^{n_i} C_j^{(i)} - \sum_{i=n_i+1}^{p_i} C_j^{(1)} > 0 \quad ... (3.6)$$

Similarly from the remaining integrals, integrals involving product of r, H-functions can also be derived but for the sake of brevity,they are not presented here. A detailed account of the H- function is available from the monograph of Mathai and Saxena [7].

(2) If we take $r = 1$; $\beta = 0$ in (3.1) it gives

$$\int_0^1 t^{\rho-1}(1-t)^{\rho}\,[1+at+b(1-t)]^{-2\rho-1}\;{}_2F_1[\nu, \delta; \tfrac{1}{2}(\nu+\delta+2);$$

$$\cdot \frac{t(1+a)}{1+at+b(1-t)}]\, H^{m_1, n_1}_{p_1, q_1}\left[z_1 R_1^1 \,\middle|\, \begin{matrix}(c_j, C_j)_{1,p_1} \\ (d_j, D_j)_{1,q_1}\end{matrix}\right] dt$$

$$= \frac{2^{\nu+\delta-2\rho-1}\,\Gamma(\frac{\nu}{2}+\frac{\delta}{2}+1)}{(\nu-\delta)(1+a)^{\rho}\,(1+b)^{\rho+1}\,\Gamma(\nu)\,\Gamma(\delta)}\{\Gamma(\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\delta}{2})$$

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$$\cdot H^{m_1, n_1+3}_{p_1+3, q_1+3}\left[z_1\left|\begin{matrix} I_1'' \\ I_2'' \end{matrix}\right.\right] - \Gamma(\frac{\nu}{2})\,\Gamma(\frac{\delta}{2}+\frac{1}{2})\,H^{m_1, n_1+3}_{p_1+3, q_1+3}\left[z_1\left|\begin{matrix} J_1'' \\ J_2'' \end{matrix}\right.\right] \quad \ldots (3.7)$$

where

$$R_1 = \frac{4(t+at)(1+b)(1-t)}{[1+at+b(1-t)]^2}$$

$$I_1'' = (1-\rho, \lambda_1); (1-\rho+\frac{\nu}{2}+\frac{\delta}{2}; \lambda_1); (\frac{\nu}{2}-\frac{\delta}{2}-\rho, \lambda_1)(c_j, C_j)_{1, p_1} \quad \ldots . (3.8)$$

$$I_2'' = (d_j, D_j)_{1, q_1}; (1-\rho+\frac{\nu}{2}-\frac{\delta}{2}, \lambda_1); (\frac{\nu}{2}-\rho, \lambda_1); (\frac{1}{2}+\frac{\delta}{2}-\rho, \lambda_1) \quad \ldots (3.9)$$

$$J_1'' = (1-\rho, \lambda_1), (1-\rho+\frac{\nu}{2}+\frac{\delta}{2}; \lambda_1); (\frac{\delta}{2}, -\frac{\nu}{2}-\rho, \lambda_1)(c_j, C_j)_{1, p_1} \quad \ldots . (3.10)$$

$$J_2'' = (d_j, D_j)_{1, q_1}; (1-\rho-\frac{\nu}{2}+\frac{\delta}{2}, \lambda_1); (\frac{\delta}{2}-\rho, \lambda_1); (\frac{1}{2}+\frac{\nu}{2}-\rho, \lambda_1) \quad \ldots (3.11)$$

The (sufficient) conditions of validity of (3.7) are given below:

(i) The constants a and b are such that none of the expressions $1+a$, $1+b$, $1+at+b(1-t)$ where $0 \le t \le 1$ is zero.

(ii) $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(2\rho - \nu - \delta) > 0$.

(iii) $\mathrm{Re}(\rho + \lambda_1 \min_{1 \le j \le m_1} [\mathrm{Re}(d_j/D_j]) > 0; \lambda_1 \ge 0.$

(iv) $\Omega > 0, \quad |\arg z_1) < \frac{1}{2}\pi\Omega.$

where $\Omega = \sum_{j=1}^{n_1} C_j - \sum_{j=n_1+1}^{p_1} C_j + \sum_{j=1}^{m_1} D_j - \sum_{j=m_1+1}^{q_1} D_j > 0.$

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COMMON FIXED POINTS OF WEAKLY COMMUTING MAPPINGS

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ABSTRACT

A common fixed point theorem for two sequences of self-mappings, respectively weakly commuting with two given self-mapping of a complete metric space and satisfying a Meir and Keeler type contractive condition is obtained.The existence of the fixed point has been established assuming the continuity of only one of the mappings. Our work generalizes several well known results on contractive mappings.

1. INTRODUCTION

The study of common fixed points of contractive type mappings has emerged as an area of vigorous research activity and a number of interesting results have been reported.Majority of these results either deal with commuting mappings or with more generalized concept of weak commutativity of mappings introduced by Sessa [8]. Jungeck [2] introduced the notion of compatibility of mappings, also called asymptotic commutativity by Tivari and Singh[9] in an independent formulation. It was claimed that weak commutativity implies compatibility [3], [9] but not conversely [7]. However, in a review of [3] (Mathematical Review 89 h : 54030) Singh has shown the existence of a weakly commuting pair of mappings satisfying a contractive condition for which there exists no sequence of points satisfying the condition of compatibility.

In this paper we obtain a common fixed point theorem for two sequences of selfmappings respectively weakly commuting with two selfmappings and satisfying a Meir and Keeler type contractive condition. The theorem assumes the continuity of only one of the mappings. The mapping condition studied by us is a generalization of the mapping condition 22 of Rhoades [6]. Our work generalizes the results due to Fisher [1], Pant [4], Park and Rhoades [5] and a number of other results.

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2. RESULTS

If (X, d) be a metric space, two selfmappings F and G of X are called weakly commuting provided $d(FGx, GFx) \leq d(Fx, Gx)$ for each x in X.

Theorem. Let $\{P_i\}$ and $\{Q_j\}$, $i, j = 1, 2, 3, \ldots$, be sequences of selfmappings and let S and T be selfmappings of a complete metric space (X, d) satisfying the conditions :

Given $\varepsilon > 0$, there exists an $h(\varepsilon) > 0$, $h(\varepsilon)$ being a nondecreasing function of ε, such that for all x, y in X

$$\varepsilon \leq \max \{d(Sx, Ty), d(P_i x, Sx), d(Q_j y, Ty), [d(P_i x, Ty) + d(Q_j, Sx)]/2\} < \varepsilon + h$$

$$\Rightarrow \quad d(P_i x, Q_j y) < \varepsilon, \qquad \ldots (1)$$

$$P_i x = Q_j y \quad \text{whenever} \quad P_i x = Sx, \; Q_j y = Ty. \qquad \ldots (2)$$

Let the range of T contain the range of each P_i and the range of S contain the range of each Q_j. If each P_i weakly commutes with S and each Q_j weakly commutes with T and if one of the mappings $\{P_i\}$, $\{Q_j\}$, S or T be continuous then the mappings $\{P_i\}$, $\{Q_j\}$, S and T, $i, j = 1, 2, 3, \ldots$, have a unique common fixed point which is also the unique common fixed point of P_i and S and of Q_j and T.

Proof. First, with the help of (1), we note that for all x, y in X such that $P_i x \neq Sx$, $Q_j y \neq Ty$, $i, j = 1, 2, 3, \ldots$,

$$d(P_i x, Q_j y) < \max\{d(Sx, Ty), d(P_i x, Sx), d(Q_j y, Ty), [d(P_i x, Ty) + d(Q_j y, Sx)]/2\}. \qquad \ldots (3)$$

Secondly, the nondecreasing character of $h(\varepsilon)$ implies that given $\varepsilon > 0$, there exists $\varepsilon_0 > 0$, such that $\varepsilon_0 < \varepsilon < \varepsilon_0 + h(\varepsilon_0)$ or equivalently

$$\max \{d(Sx, Ty), d(P_i x, Sx), d(Q_j y, Ty), [d(P_i x, Ty) + d(Q_j y, Sx)]/2\} = \varepsilon$$

$$\Rightarrow \quad d(P_i x, Q_j y) < \varepsilon_0, \; \varepsilon_0 < \varepsilon. \qquad \ldots (4)$$

Let us arbitrarily select a pair of integers i and j and let x_0 be any point in X. Choose a sequence of points $\{x_n : n = 0, 1, 2, \ldots\}$ in X defined by $P_i x_{2n} = T x_{2n+1}$ and $Q_j x_{2n+1} = S x_{2n+2}$. This can be done since the ranges of T and S respectively contain the ranges of P_i and Q_j. We can assume that $P_i x_{2n} \neq Q_j x_{2n+1}$ and $Q_j x_{2n+1} \neq P_i x_{2n+2}$ for every value of n, otherwise the existence of the fixed point is easy to establish. Then from (3) we obtain

$$d(P_i x_{2n}, Q_j x_{2n+1}) < d(Q_j x_{2n-1}, P_i x_{2n}) \qquad \ldots (5)$$

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and $d(Q_j x_{2n-1}, P_i x_{2n}) < d(P_i x_{2n-2}, Q_j x_{2n-1})$. ... (6)

Similarly, for every integer $p > 0$

$$d(Q_j x_{2n+1}, P_i x_{2(n+p)+2}) < d(P_i x_{2n}, Q_j x_{2(n+p)+1}) + d(P_i x_{2n}, Q_j x_{2n+1}) \quad \ldots (7)$$

and $$d(P_i x_{2n}, Q_j x_{2(n+p)+1}) < d(Q_j x_{2n-1}, P_i x_{2(n+p)}) + d(Q_j x_{2n-1}, P_i x_{2n}). \quad \ldots (8)$$

In view of (5) and (6) we claim that $\lim_{n\to\infty} d(P_i x_{2n}, Q_j x_{2n+1}) = 0 = \lim_{n\to\infty} d(Q_j x_{2n+1}, P_i x_{2n+2})$. For, if not, suppose for instance $\lim_{n\to\infty} d(P_i x_{2n}, Q_j x_{2n+1}) = r$, $r > 0$. Then given $h > 0$, there exists a positive integer N_1, such that for each integer $m \geq N_1$ we have

$$r \leq d(P_i x_{2m}, Q_j x_{2m+1}) < r + h \quad \ldots (9)$$

or $$r \leq \max \{d(Sx_{2m+2}, Tx_{2m+1}), d(P_i x_{2m+2}, Sx_{2m+2}), d(Q_j x_{2m+1}, Tx_{2m+1}),$$
$$[d(P_i x_{2m+2}, Tx_{2m+1}) + d(Q_j x_{2m+1}, Sx_{2m+2})/2\} < r + h. \quad \ldots (10)$$

Selecting h in (10) in accordance with (1), for each integer $m \geq N_1$, we obtain $d(P_i x_{2m+2}, Q_j x_{2m+1}) < r$ and so $d(P_i x_{2m+2}, Q_j x_{2m+3}) < r$, which contradicts (9). Therefore

$$\lim_{n\to\infty} d(P_i x_{2n}, Q_j x_{2n+1}) = 0 = \lim_{n\to\infty} d(Q_j x_{2n+1}, P_i x_{2n+2}). \quad \ldots (11)$$

Also by virtue of inequalities (7), (8) and equation (11), it follows that

$$\lim_{n\to\infty} d(P_i x_{2n}, Q_j x_{2(n+p)+1}) = \lim_{n\to\infty} d(Q_j x_{2n+1}, P_i x_{2(n+p)+2}).$$

Now,if possible, suppose $\lim_{n\to\infty} d(P_i x_{2n}, Q_j x_{2(n+p)+1}) = r$, $r > 0$.

Then in view of this and equation (11), given $h > 0$ there exists a positive integer N_2 such that for each integer $m \geq N_2$, we have

$$r \leq d(P_i x_{2m}, Q_j x_{2(m+p)+1}) = d(Tx_{2m+1}, Sx_{2(m+p)+2}) < r + h/4, \quad \ldots (12)$$

$$r \leq d(Q_j x_{2m+1}, P_i x_{2(m+p)+2}) = d(Sx_{2m+2}, Tx_{2(m+p)+3}) < r + h/4, \quad \ldots (13)$$

$$d(P_i x_{2m}, Q_j x_{2m+1}) < h/4$$

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and $d(Q_j x_{2m-1}, P_i x_{2m}) < h/4$.

Therefore for each $m \geq N_2$ we have

$$r \leq \max \{d(Sx_{2m+2}, Tx_{2(m+p)+3}), d(P_i x_{2m+2}, Sx_{2m+2}), d(Q_j x_{2(m+p)+3}, Tx_{2(m+p)+3}), [d(P_i x_{2m+2}, Tx_{2(m+p)+3}) + d(Q_j x_{2(m+p)+3}, Sx_{2m+2})]/2)\} < r + h/4. \quad \ldots (14)$$

Selecting h in (14) in accordance with (1), for each $m \geq N_2$, we have $d(P_i x_{2m+2}, Q_j x_{2(m+p)+3}) < r$, which contradicts (12). Hence $\{P_i x_0, Q_j x_1, \ldots, P_i x_{2n}, Q_j x_{2n+1}, \ldots\}$ is a Cauchy sequence in the complete metric space X and so has a limit point z in X. Also the sequences $\{P_i x_{2n}\} = \{Tx_{2n+1}\}$ and $\{Q_j x_{2n+1}\} = \{Sx_{2n+2}\}$ converge to the point z.

Let us now suppose that the mapping S is continuous. Then since P_i and S commute weakly, the sequences $\{P_i S x_{2n}\}$ and $\{SSx_{2n}\}$ converge to Sz. We claim that $z = Sz$. For if $z \neq Sz$, then the inequality

$$d(P_i Sx_{2n}, Q_j x_{2n+1}) < \max \{d(SSx_{2n}, Tx_{2n+1}), d(P_i Sx_{2n}, SSx_{2n}), d(Q_j x_{2n+1}, Tx_{2n+1}), [d(P_i Sx_{2n}, Tx_{2n+1}) + d(Q_j x_{2n+1}, SSx_{2n})]/2\}$$

on letting $n \to \infty$ and in view of (4) leads to $d(z, Sz) < d(z, Sz)$, contradiction. Therefore $z = Sz$. Similarly the inequality

$$d(P_i z, Q_j x_{2n+1}) < \max \{d(Sz, Tx_{2n+1}), d(P_i z, Sz), d(Q_j x_{2n+1}, Tx_{2n+1}), [d(P_i z, Tx_{2n+1}) + d(Q_j x_{2n+1}, Sz)]/2\}$$

yields $z = P_i z$. This means that there exists a point z_0 *in* X such that $Sz = z = P_i z = Tz_0$, as the range of P_i is contained in the range of T. Morevoer, the inequality.

$$d(P_i z, Q_j z_0) \max \{d(Sz, Tz_0), d(P_i z, Sz), d(Q_j z_0, Tz_0), [d(P_i z, Tz_0) + d(Q_j z_0, Sz)]/2\}$$

yields $z = Q_j z_0 = Tz_0$. This equation, in view of weak commutativity of Q_j and T, implies $Q_j z = Tz$. Finally $Q_j z = Tz$ together with (3) leads to $z = Q_j z = Tz$.

We have therefore proved that z is a common fixed point of P_i, Q_j, S and T for an arbitrarily chosen pair of integers i and j.

Now, if possible suppose w is a second common fixed point of P_i and S. Then

$$d(w, z) = d(P_i w, Q_j z) < \max \{d(Sw, Tz), d(P_i w, Sw), d(Q_j z, Tz),$$

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$$[d(P_i w, Tz) + d(Q_j z, Sw)]/2\}$$
$$= d(w, z),$$

a contradiction. Hence z is the unique common fixed point of P_i and S. Similarly z is also the unique common fixed point of Q_j and T.

Now if we keep i fixed and vary j, we find that z is the unique common fixed point of Q_j and T for every value of j. On the other hand, since $z = Q_j z = Tz$, if we keep j fixed and vary i it can be easily shown that z is the unique common fixed point of P_i and S for every value of i.

Using similar type of arguments we can prove that z is the unique common fixed point of P_i, Q_j, S and T if. P_i is assumed to be continuous, for some value of i, instead of S.

Similarly z is the unique common fixed point of P_i, Q_j, S and T if either T or Q_j is continuous instead of S or P_i. respectively.

This completes the proof of the theorem.

Remark 1. In the above theorem if we take $P_i = P = Q_j$ for each i and each j and $S = T$, we obtain Theorem 4 of Park and Rhoades [5] as a particular case of our theorem. In that case we can drop the assumption on nondecreasing character of $h(\varepsilon)$

Remark 2. In this theorem if we take $P_i = P$ and $Q_j = Q$ for each i and each j and if we take $\max\{d(Sx, Ty, d(Px, Sx), d(Qy, Ty), [d(Px, Ty) + d(Qy, Sx]/2\} = \max\{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\}$, we obtain the Theorem of Pant [4]. Further by choosing $h(\varepsilon) = 2(1 - c)\,\varepsilon/c$, $0 < c < 1$, we shall obtain Theorem 2 of Fisher [1] as a special case.

We now give an example of mappings satisfying the conditions of the above theorem and having a unique common fixed point.

Example. Let $X = [0, 1]$ and let d be the usual metric on X. Define selfmappings P_i, Q_j, S and T on $X, i, j = 1, 2, 3, \ldots$, as follows

$P_i x = ix/(6i + 1)$, $\quad x \neq 1$, $\quad P_i 1 = i/(12i + 1)$

$Q_j x = 0$, $\quad x \neq 1$, $\quad Q_j 1, = j/(6j + 1)$

$Sx = x/2$, $\quad x \neq 1$, $\quad S1 = 1/4$

$Tx = x$ for each x in X.

Then $\{P_i\}$, $\{Q_j\}$, S and T satisfy all the conditions of the theorem and have a unique common fixed point $x = 0$.

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COMMON FIXED POINTS OF TWO PAIRS OF SEQUENCES OF MAPPINGS

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ABSTRACT

A common fixed point theorem for two pairs of sequences of pairwise weakly commuting selfmappings of a complete metric space satisfying Meir and Keeler type contractive condition is obtained. The existence of the fixed point has been established under the assumption of continuity of only one of the mappings. Our work generalizes several fixed point theorems concerning contractive mappings.

1. INTRODUCTION

The study of common fixed points of mappings satisfying some contractive type condition has been at the centre of vigorous research activity and a number of interesting results have been obtained by various authors. Most of these results either deal with commuting mappings or assume the notion of weak commutativity of mappings introduced by Sessa [8]. Jungck [2] introduced the notion of compatibility of mappings, also called asymptotic commutativity by Tivari and Singh [9] in an independent formulation. It was claimed that weak commutativity implies compatibility [3], [9] but not conversely [7]. However, in a review of [3] (Mathematical Review 89 h : 54030) Singh has produced an example to show the existence of a weakly commuting pairs of mappings satisfying a contractive condition for which there exists no sequence of points satisfying the condition of compatibility.

In this paper we obtain a common fixed point theorem for two pairs of sequences of pairwise weekly commuting mappings satisfying a Meir and Keeler type contractive condition. We have assumed the continuity of only one of the mappings. The mapping condition studied by us is a generalization of mapping condition 10 of Rhoades [6]. Our work generalizes the results due to Fisher [1], Pant [4], Park and Bae [5] and some other results.

2. RESULTS

If (X, d) be a metric space, two selfmappings F and G of X are called weakly commuting provided $d(FGX, GFx) \leq d(Fx, Gx)$ for each x in X.

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Theorem. Let $\{P_i\}, \{S_i\}, \{Q_j\}$ and $\{T_j\}$, $i, j = 1, 2, 3, \ldots$, be sequences of mapings of a complete metric space (X, d) into itself satisfying the conditions :

Given $\varepsilon > 0$, there exists an $h(\varepsilon) > 0$, $h(\varepsilon)$ being nondecreasing function of ε, such that for all x, y in X

$$\varepsilon \le \max \{d(S_i x, T_j y), d(P_i x, S_i x), d(Q_j y, T_j y)\} < \varepsilon + h$$

$$\Rightarrow \quad d(P_i x, Q_j y) < \varepsilon \qquad \ldots (1)$$

$$P_i x = Q_j y \quad \text{whenever } P_i x = S_i x, Q_j y = T_j y. \qquad \ldots (2)$$

Let P_i and S_i commute weakly and Q_j and T_j commute weakly for every value of i and j respectively. If the ranges of T_j and S_i respectively contain the ranges of P_i and Q_j for every value if i and j and if one of the mappings is continuous then the sequences $\{P_i\}, \{S_i\}, \{Q_j\}$ and $\{T_j\}$ have a unique common fixed point which is also the unique common fixed point of $\{P_i\}$ and $\{S_i\}$ and of the sequences $\{Q_j\}$ and $\{T_j\}$.

Proof. First, with the help of (1), we note that for all x, y in X such that $P_i x \ne S_i x$, $Q_j y \ne T_j y$, $i, j = 1, 2, 3, \ldots$,

$$d(P_i x, Q_j y) < \max \{d(S_i x, T_j y), d(P_i x, S_i x), d(Q_j y, T_j y)\}. \qquad \ldots (3)$$

Secondly, the nondecreasing character of $h(\varepsilon)$ implies that given $\varepsilon > 0$, there exists $\varepsilon_0 > 0$, such that $\varepsilon_0 < \varepsilon < \varepsilon_0 + h(\varepsilon_0)$ or equivalently.

$$\max\{d(S_i x, T_j y), d(P_i x, S_i x), d(Q_j y, T_j y)\} = \varepsilon$$

$$\Rightarrow \quad d(P_i x, Q_j y) < \varepsilon_0, \varepsilon_0 0 < \varepsilon. \qquad \ldots (4)$$

Let us arbitrarily select a pair of integers i and j and let x_0 be any point in X, Choose a sequence of points $\{x_n : n = 0, 1, 2, \ldots\}$ in X defined by $P_i x_{2n} = T_j x_{2n+1}$ and $Q_j x_{2n+1} = S_i x_{2n+2}$. This can be done since the ranges of T_j and S_i respectively contain the ranges of P_i and Q_j. We can assume that $P_i x_{2n} \ne Q_j x_{2n+1}$ and $Q_j x_{2n+1} \ne P_i x_{2n+2}$ for every value of n, otherwise the existence of the common fixed point is easy to establish. Then from (3), we obtain

$$d(P_i x_{2n}, Q_j x_{2n+1}) < d(Q_j x_{2n-1}, P_i x_{2n}), \qquad \ldots (5)$$

and $$d(Q_j x_{2n-1}, P_i x_{2n}) < d(P_i x_{2n-2}, Q_j x_{2n-1}). \qquad \ldots (6)$$

Also for any integer $p > 0$

$$d(Q_j x_{2n+1}, P_i x_{2(n+p)+2}) < \max \{d(P_i x_{2n}, Q_j x_{2(n+p)+1}), d(P_i x_{2n}, Q_j x_{2n+1})\} \qquad \ldots (7)$$

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and $d(P_i x_{2n}, Q_j x_{2(n+p)+1})$

$$< \max \{d(Q_j x_{2n-1}, P_i x_{2n}), d(Q_j x_{2n-1}, P_i x_{2(n+p)})\}. \quad \ldots (8)$$

In view of (5) and (6) we claim that $\lim_{n \to \infty} d(P_i x_{2n}, Q_j x_{2n+1}) = 0 = \lim_{n \to \infty} d(Q_j x_{2n+1}, P_i x_{2n+2})$. For, if not, suppose for instance $\lim_{n \to \infty} d(P_i x_{2n}, Q_j x_{2n+1}) = r$, $r > 0$. Then given $h > 0$ there exists a positive integer N such that for each integer $m \geq N$, we have

$$r \leq d(P_i x_{2m}, Q_j x_{2m+1}) < r + h \quad \ldots (9)$$

or $r \leq \max \{d(S_i x_{2m+2}, T_j x_{2m+1}), d(P_i x_{2m+2}, S_i x_{2m+2}),$

$$d(Q_j x_{2m+1}, T_j x_{2m+1})\} < r + h. \quad \ldots (10)$$

Selecting h in (10) in accordance with (1), for each $m \geq N$ we obtain $d(P_i x_{2m+2}, Q_j x_{2m+1}) < r$ and so $d(P_i x_{2m+2}, Q_j x_{2m+3}) < r$, which contradicts (9). Therefore

$$\lim_{n \to \infty} d(P_i x_{2n}, Q_j x_{2n+1}) = 0 = \lim_{n \to \infty} d(Q_j x_{2n+1}, P_i x_{2n+2}) \quad \ldots (11)$$

Also, in view of (7) (8) and (11), it follows that

$$\lim_{n \to \infty} d(P_i x_{2n}, Q_j x_{2(n+p)+1}) = \lim_{n \to \infty} d(Q_j x_{2n+1}, P_i x_{2(n+p)+2}).$$

Following similar argument, as used to evaluate limit (11),it follows easily that

$$\lim_{n \to \infty} d(P_i x_{2n}, Q_j x_{2(n+p)+1}) = 0 = \lim_{n \to \infty} d(Q_j x_{2n+1}, P_i x_{2(n+p)+2}).$$

Hence $\{P_i x_0, Q_j x_1, \ldots, P_i x_{2n}, Q_j x_{2n+1}, \ldots\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X.

Also, the sequence $\{P_i x_{2n} = T_j x_{2n+1}\}$ and $\{Q_j x_{2n+1} = S_i x_{2n+2}\}$ converge to z.

Let us now suppose that the mapping S_i is continuous. Then, since P_i and S_i commute weakly, the sequences $\{P_i S_i x_{2n}\}$ and $\{S_i S_i x_{2n}\}$ coverage to $S_i z$. We claim that $z = S_i z$. For if $z \neq S_i z$ then the inequality

$$d(P_i S_i x_{2n}, Q_j x_{2n+1}) < \max \{d(S_i S_i x_{2n}, T_j x_{2n+1}),$$

$$d(P_i S_i x_{2n}, S_i S_i x_{2n}), d(Q_j x_{2n+1}, T_j x_{2n+1})\}$$

on letting $n \to \infty$ and in view of (4) leads to $d(z, S_i z) < d(z, S_i z)$, a contradiction. Therefore $z = S_i z$. Similarly the inequality

$$d(P_i z, Q_j x_{2n+1}) < \max\{d(S_i z, T_j x_{2n+1}), d(P_i z, S_i z),$$

$$d(Q_j z_{2n+1}, T_j x_{2n+1})\}$$

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yields $z = P_i z$. This means that there exists a point z_0 in X such that $S_i z = z = P_i z = T_j z_0$, as the range of P_i is contained in the range of T_j. Moreover, the inequality

$$d(P_i z, Q_j z_0) < \max \{d(S_i z, T_j z_0), d(P_i z, S_i z), d(Q_j z_0, T_j z_0)\}$$

yields $z = Q_j z_0 = T_j z_0$. This equation, in view of weak commutativity of Q_j and T_j, implies $Q_j z = T_j z$. Finally, the relation $Q_j z = T_j z$ together with (3) leads to $z = Q_j z = T_j z$.

We have therefore proved that z is a common fixed point of P_i, Q_j, S_i and T_j for the arbitrarily chosen pair of integers i and j.

Now, if possible, suppose w is a second common fixed point of P_i and S_i. Then

$$\begin{aligned} d(w, z) &= d(P_i w, Q_j z) \\ &< \max \{d(S_i w, T_j z), d(P_i w, S_i w), d(Q_j z, T_j z)\} \\ &= d(w, z), \end{aligned}$$

a contradiction. Hence z is the unique common fixed point of P_i and S_i. Similarly z is the unique common fixed point of Q_j and T_j.

Now, if we keep i fixed and vary j, we find that z is the unique common fixed point of Q_j and T_j for every value of j. On the other hand, since $Q_j z = z = T_j z$, if we keep j fixed and vary i and also drop the assumption of continuity of S_i we can easily show that z is the unique common fixed point of P_i and S_i for every value of i.

Using similar type of arguments we can prove that z is the unique common fixed point of P_i, Q_j, S_i, and T_j if P_i is assumed to be continuous instead of S_i.

Similarly z is the unique common fixed point of P_i, Q_j, S_i and T_j if either Q_j or T_j is continuous instead of S_i or P_i respectively.

This completes the proof of the theorem.

Remark 1. In the above theorem if we take

$P_i = P$, $Q_j = Q$, $S_i = AS$, and $T_j = T$ for each i and j, we obtain the theorem of Pant [2].

Remark 2. In the above theorem if we take $P_i = P$, $Q_j = Q$, $S_i = S$ and $T_j = T$ for each i and j and let $h(\varepsilon) = 2(1 - c)\, \varepsilon/c$, $0 < c < 1$, then we obtain Theorem 2 of Fisher [1].

Remark 3. In the above theorem if we take $P_i = Q_j = g$ and $S_i = T_j = f$ for every value of i and j and let

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$$\max \{d(S_i x, T_j y), d(P_i x, S_i x), d(Q_j y, T_j y)\} = d(S_i x, T_j y),$$

then we can drop the assumption on nondecreasing character of $h(\varepsilon)$ and we obtain the Theorem 2.4 of Park and Bae [3].

We now give an example of sequences of mappings satisfying the conditions of the above theorem and having a unique common fixed point.

Example. Let $X = [0, 1]$ and let d be the usual metric on X. Define selfmappings P_i, Q_j, S_i and T_j on $X, i, j = 1, 2, 3, \ldots$, as follows

$$P_i x = ix/(6i + 1), \quad x \neq 1, \quad P_i 1 = i/(12i + 1)$$

$$Q_j x = 0, \quad x \neq 1, \quad Q_j 1 = j/(6j + 1)$$

$$S_i x = ix/(2i + 1), \quad x \neq 1, \quad S_i 1 = i/(4i + 1)$$

$$T_j x = x \text{ for each } x \text{ in } X.$$

Then $\{P_i\}$, $\{Q_j\}$, $\{S_i\}$ and $\{T_j\}$ satisfy all the conditions of the theorem and have a unique common fixed point $x = 0$.

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INTEGRALS INVOLVING A GENERAL CLASS OF MULTIVARIABLE POLYNOMIALS, JACOBI POLYNOMIALS, AND FOX'S *H*-FUNCTION

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ABSTRACT

Here we evaluate four integrals involving various products of general class of multivariable polynomials introduced earlier by H.M. Srivastava and M. Garg, Jacobi ploynomials, and Fox's *H*- function. Our results are quite general in character and provide interesting unifications and generalizations of a large number of (known or new) integral formulae.

1. Introduction

Recently Kalla, Conde and Luke [2], Kalla [3] and Kant and Koul [4] have established certain integrals involving Jacobi polynomials, generalized Jacobi functions, a general class of polynomials and Fox's *H*-function. In an attempt to unify and extend these results we have evaluated four finite integrals involving the product of Jacobi polynomials, a general class of multivariable polynomials and Fox's *H*-function. The technique followed is essentially that of Kalla et al.([2],[3]).

The general class of polynomials $S_n^m[x]$ introduced by Srivastava [7],p.1, Eq. (1)] has further been generalized to a multivariable polynomial in the following manner (Srivastava and Garg [8] :

$$S_n^{m_1, \ldots, m_r}(x_1, \ldots, x_r) = \sum_{\substack{k_1, \ldots, k_r = 0 \\ m_1 k_1 + \ldots + m_r k_r \le n}} (-n)_{m_1 k_1 + \ldots + m_r k_r} A(n; k_1, \ldots, k_r) \frac{x_1^{k_1} \ldots x_r^{k_r}}{k_1! \ldots k_r!} \quad \ldots (1.1)$$

where $m_1, \ldots, m_r$ are arbitrary positive integers and the coefficients $A(n; k_1, \ldots, k_r)$ $(n, k_i \ge 0, i = 1, \ldots, r)$ are arbitrary constants real or complex.

To facilitate the derivation of our main integrals given in the next section we shall require an integral contained in the following

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Lemma If $\mu, \nu, \gamma_i, \delta_i$ $(i = 1, ..., r)$ are all non-negative real numbers (not all zero simultaneously), $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$,

$$\mathrm{Re}(\rho) + \mu \min_{1 \le j \le M} [\mathrm{Re}(b_j/B_j)] > -1, \quad \mathrm{Re}(\sigma) + \nu \min_{1 \le j \le M} [\mathrm{Re}(b_j/B_j)] > -1,$$

$A > 0$, $\delta > 0$, $|arg(z)| < \frac{1}{2} A\pi$, then

$$\int_0^t (t-x)^{\rho} x^{\sigma} p_u^{(\alpha,\beta)} (1 - z' x) S_n^{m_1, ..., m_r}(z_1 (t-x)^{\gamma_1} x^{\delta_1}, ..., z_r (t-x)^{\gamma_r} x^{\delta_r})$$

$$H_{P,Q}^{M,N}\left[z(t-x)^{\mu} x^{\nu} \middle| \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1\,Q} \end{matrix} \right] dx$$

$$= t^{\rho+\sigma+1} \sum_{\substack{m_1 k_1 + ... + m_r k_r \le n \\ k_1, ..., k_r = 0}} \sum_{w=0}^{u} \frac{(-n)_{m_1 k_1 + ... + m_r k_r} A(n; k_1, ..., k_r)}{k_1 ! ... k_r !}$$

$$\frac{\Gamma(\alpha + u + 1)\, (-u)_w\, (\alpha + \beta + u + 1)_w}{\Gamma(\alpha + w + 1)\, u !\, w !} \left(\tfrac{1}{2} z' t\right)^w \prod_{i=1}^{r} (z_i t^{\gamma_i + \delta_i})^{k_i}$$

$$H_{p+2, Q+1}^{M, N+2}\left[z t^{\mu+\nu} \middle| \begin{matrix} (-\rho - \sum_{i=1}^{r} \gamma_i k_i, \mu), (-\sigma - w - \sum_{i=1}^{r} \delta_i k_i, \nu), (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,Q}, (1 - \rho - \sigma - w - \sum_{i=1}^{r} (\gamma_i + \delta_i) k_i, \mu + \nu) \end{matrix} \right] \quad \text{.... (1.2)}$$

$$= t^{\rho+\sigma+1} \sum_{\substack{m_1 k_1 + ... + m_r k_r \le n \\ k_1, ..., k_r = 0}} \sum_{w=0}^{u} \sum_{h=0}^{M} \sum_{R=0}^{\infty} \frac{(-n)_{m_1 k_1 + ... + m_r k_r}}{k_1 ... k_r !} A(n; k_1, ..., k_r)$$

$$\prod_{i=1}^{r} (z_i t^{\gamma_i + \delta_i})^{k_i} \frac{\Gamma(\alpha + u + 1)\, (-u)_w\, (\alpha + \beta + u + 1)_w\, \Gamma(1 + \rho + \sum_{i=1}^{r} \gamma_i k_i + \mu \xi_{h,R})}{\Gamma(\alpha + w + 1)\, u !\, w !\, R !\, \Gamma(2 + \rho + \sigma + w + \sum_{i=1}^{r} (\gamma_i + \delta_i) k_i + (\mu + \nu) \xi_{h,R})}$$

$$\frac{\Gamma(1 + \sigma + w + \sum_{i=1}^{r} \delta_i k_i + \nu \xi_{h,} R)}{\Gamma(1 + \sigma + w + \sum_{i=1}^{r} \delta_i k_i + \nu \xi_{h,} R)} f(\xi_{h,R}) \left(\tfrac{1}{2} z', t\right)^w (z\, t^{\mu+\nu})^{\xi_{h,R}} \quad \text{... (1.3)}$$

$$= \sum_{\substack{m_1 k_1 + ... + m_r k_r \le n \\ k_1, ..., k_r = 0}} \sum_{w=0}^{u} \sum_{h=0}^{M} \sum_{R=0}^{\infty} g(w, h, R) \text{ ... (say)} \quad \text{... (1.4)}$$

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where $\xi_{h,R} = \dfrac{b_h + R}{B_h}$, $(R = 0, 1, 2, \ldots)$ ... (1.5)

$$f(\xi_{h,R}) = \frac{\prod_{\substack{j=1 \\ j\neq h}}^{M} \Gamma(b_j - B_j \xi_{h,R}) \prod_{j=1}^{N} \Gamma(1 - a_j - A_J \xi_{h,R})}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + B_j \xi_{h,R}) \prod_{j=N+1}^{P} \Gamma(a_j - A_j \xi_{h,R})\, B_h} \quad \ldots (1.6)$$

$$A = \sum_{j=1}^{N} A_j - \sum_{j=N+1}^{P} A_j + \sum_{j=1}^{M} B_j - \sum_{j=M+1}^{Q} B_j \quad \ldots (1.7)$$

and $\delta = \sum_{j=1}^{Q} B_j - \sum_{j=1}^{P} A_j$. ... (1.8)

The integral (1.2) is a special case of an integral recently established by the author ([6], p.10, Eq. (2.1). The value of the integral in the form given by (1.3) can be obtained easily by using the following series expansion for Fox's H-function due to Braaksma [1], see also Srivastava et al. ([9], p.12, Eq. (2..2.4))

$$H_{P,Q}^{M,N}[x] = H_{P,Q}^{M,N}\left[x \,\middle|\, \begin{matrix} (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q} \end{matrix} \right] = \sum_{h=1}^{M} \sum_{R=0}^{\infty} f(\xi_{h,R}) x^{\xi_{h,R}} \quad \ldots (1.9)$$

where $\xi_{h,R}$ and $f(\xi_{h,R})$ are defined by (1.5) and (1.6) respectively. $H_{P,Q}^{M,N}[x]$ stands for the well-known Fox's H-function. For details of this function one can refer to the book by Srivastava et al. [9].

2. Main Integrals

Let $\mu, \nu, \gamma_i, \delta_i$ are all non-negative real numbers (not all zero simultaneously), $\delta > 0, A > 0$, $|\arg(z)| < \frac{1}{2} A\pi, A$ and δ being given by (1.7) and (1.8) , respectively.

$$\mathrm{Re}(\rho) + \mu \min_{1 \le j \le M} \{\mathrm{Re}(b_j / B_j)\} > -1$$

$$\mathrm{Re}(\sigma) + \nu \min_{l \le j \le M} \{\mathrm{Re}(b_j / B_j)\} > -1,$$

and

$$F(x) = (t-x)^{\rho} x^{\sigma} P_u^{(\alpha,\beta)}(1 - z' x)\, S_n^{m_1, \ldots, m_r}(z_1 (t-x)^{\gamma_1} x^{\delta_1}, \ldots,$$

$$z_r (t-x)^{\gamma_r} x^{\delta_r})\, H_{P,Q}^{M,N}\left[z(t-x)^{\mu} x^{\nu} \,\middle|\, \begin{matrix} (a_j, A_j)_{1,P} \\ b_j, B_j)_{1,Q} \end{matrix} \right] \quad \ldots (2.1)$$

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then

$$\text{(i)} \int_0^t F(x) \log (t-x)\, dx = \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \ldots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=1}^{M} \sum_{R=0}^{\infty} g(w, h, R)$$

$$\left[\log t + \psi\left(1 + \rho + \sum_{i=1}^{r} \gamma_i k_i + \mu \xi_{h, r}\right) \psi\left(2 + \rho + \sigma + w + \sum_{i=1}^{r} (\gamma_i + \delta_i) k_i + (\mu + \nu)\xi_{h, R}\right)\right] \quad \ldots (2.2)$$

$$\text{(ii)} \int_0^t F(x) \log x\, dx = \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \ldots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=1}^{M} \sum_{R=0}^{\infty} g(w, h, R).$$

$$\left[\log t + \psi\left(1 + \sigma + \sum_{i=1}^{r} \delta_i k_i + \nu \xi_{h, R}\right) - \psi\left(2 + \rho + 2 + w + \sum_{i=1}^{r} (\gamma_i + \delta_i) k_i + (\mu + \nu)\xi_{h, R}\right)\right] \quad \ldots (2.3)$$

$$\text{(iii)} \int_0^t F(x) \log [(t-x)x]\, dx = \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \ldots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=1}^{M} \sum_{R=0}^{\infty} g(w, h, R)$$

$$\left[\log t^2 + \psi\left(1 + \rho + \sum_{i=1}^{r} \gamma_i k_i + \mu\, \xi_{h, R}\right) + \psi\left(1 + \sigma + w + \sum_{i=1}^{r} \delta_i k_i + \nu \xi_{h, r}\right) - 2\psi\left(2 + \rho + \sigma + w + \sum_{i=1}^{r} (\gamma_i + \delta_i) k_i + (\mu + \nu)\, \xi_{h, R}\right)\right] \quad \ldots (2.4)$$

$$\text{(iv)} \int_0^t F(x) \log \left[\frac{t-x}{x}\right] dx = \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \ldots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=1}^{M} \sum_{R=0}^{\infty} g(w, h, R)$$

$$\left[\psi\left(1 + \rho + \sum_{i=1}^{r} \gamma_i k_i + \mu\, \xi_{h, R}\right) - \psi\left(1 + \sigma + w + \sum_{i=1}^{r} \delta_i k_i + \nu \xi_{h, R}\right)\right] \quad \ldots (2.5)$$

where $\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$ and $g(w, h, R)$ and $\xi_{h, R}$ are given by (1.4) and (1.5) respectively. Also the series occurring on the right-hand side of integrals (2.2) through (2.5) converge absolutely.

Proof : The results in (2.2) and (2.3) can be obtained by taking the partial derivatives of both sides of (1.3) with respect to ρ and σ. respectively. The integral (2.4) is obtained by adding (2.2) and (2.3) and the integral (2.5) is obtained by subtracting (2.3) from (2.2).

3. Particular Cases

At the outset, we should remark that our integral formulae are quite general in character. Indeed, these results can suitably be

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specialized to a number of known and new integrals involving a large spectrum of various classes of polynomials and elementary special functions. However, we mention here only a few special cases of (2.2).

(i) If we put

$$A(n; k_1, \ldots, k_r) = \frac{\prod_{j=1}^{E} (e_j)_{k_1\theta_j' + \ldots + k_r\theta_j^{(r)}} \prod_{j=1}^{U'} (u_j')_{k_1\phi_j'} \ldots \prod_{j=1}^{U^{(r)}} (u_j^{(r)})_{k_r\varphi_j^{(r)}}}{\prod_{j=1}^{D} (d_j)_{k_1\tau_j' + \ldots + k_r\tau_j^{(r)}} \prod_{j=1}^{V'} (v_j')_{k_1\zeta_j'} \ldots \prod_{j=1}^{V^{(r)}} (v_j^{(r)})_{k_r\zeta_j^{(r)}}} \quad \ldots (3.1)$$

in (2.1) we arrive easily at the following interesting integral formula

$$\int_0^t (t-x)^{\rho} x^{\sigma} P_u^{(\alpha, \beta)} (1 - z'x) \log (t-x) \, F_{D : V', \ldots, V^{(r)}}^{1+E : U', \ldots, U^{(r)}} \left[\begin{matrix} [-n : m_1, \ldots, m_r], \\ [(e) : \theta', \ldots, \theta^{(r)}] : [(U') : \varphi'] ; \ldots ; [(U^{(r)}) : \varphi^{(r)}] \\ [(d) : \tau', \ldots, \tau^{(r)}] : [(V') : \zeta'] : \ldots : [(V^{(r)}) : \zeta^{(r)}] \end{matrix} \right.$$

$$\left. z_1 (t-x)^{\gamma_1} x^{\delta_1}, \ldots, z_r (t-x)^{\gamma_r} x^{\delta_r} \right] H_{P\,Q}^{M\,N} \left[z(t-x)^{\mu} x \left| \begin{matrix} (a_j, A_j)_{1, p} \\ (b_j, B_j)_{1, Q} \end{matrix} \right. \right] dx$$

$$= \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \ldots + m_r k_r \le n} \sum_{w=0}^{u} \sum_{h=1}^{M} \sum_{R=0}^{\infty} n(w, h, R) \left[\log t + \right.$$

$$\left. \psi \left(1 + \rho + \sum_{i=1}^{r} \gamma_i k_i + \mu \xi_{h, R}\right) - \psi \left(2 + \rho + \sigma + w + \sum_{i=1}^{r} (\gamma_i + \delta_i) k_i + (\mu + \nu) \xi_{h, R}\right) \right] \quad \ldots (3.2)$$

where $\eta(w, h, R)$ is obtained by putting the value of $A(n ; k_1, \ldots, k_r)$ from (3.1) in $g\ (w,\ h,\ R)$.

(ii) Again if we take $m_i = 0, z_i \to 0\ (i = 2, \ldots, r)$ (2.2) and replace $A(n; k_1, 0, \ldots, 0)$ by $A_{n, k}$ therein, we easily arrive at the following integral

$$\int_0^t (t-x)^{\rho} x^{\sigma} P_u^{(\alpha, \beta)} (1 - z'x) \, S_n^m [z_1 (t-x)^{\gamma} x^{\delta}] \log (t-x)$$

$$H_{P\,Q}^{(M, N)} \left[z(t-x)^{\mu} x^{\nu} \left| \begin{matrix} (a_j, A_j)_{1, P} \\ (b_j, B_j)_{1, Q} \end{matrix} \right. \right] dx$$

$$= \sum_{k=0}^{[n/m]} \sum_{w=0}^{u} \sum_{h=0}^{M} \sum_{R=0}^{\infty} t^{\rho + \sigma + 1} \frac{(-n)_{mk} A_{n, k} \, \Gamma(\alpha + u + 1)(-u)_w (\alpha + \beta + u + 1)_w}{k \,!\, \Gamma(\alpha + w + 1)\, u \,!\, w \,!\, R \,!}$$

$$\frac{\Gamma(1 + \rho + \gamma k + \mu \xi_{h, R}) \, \Gamma(1 + \sigma + w + \delta k + \nu \xi_{h, R'})}{(2 + \rho + \sigma + w + (\gamma + \delta) k + (\mu + \nu) \xi(h, R)} \left[\log t + \psi (1 + \rho + \gamma k + \mu \xi_{h, R}) \right.$$

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$$- \psi(2+\rho+\sigma+w+(\gamma+\delta)k+(\mu+\nu)\,\xi_{h,R})\Big] f(\xi_{h,R})(z_1 t^{\gamma+\delta})^k\,(\tfrac{1}{2}z't)^{w}(zt^{\mu+\nu})^{\xi_{h,R}} \qquad \ldots (3.3)$$

where

$$S_n^m(x) = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k \qquad \ldots (3.4)$$

is a general class of polynomials introduced by Srivastava ([7], p.1, Eq. (1). This general class of polynomials include almost all the wellknown orthogonal polynomial and generalized hypergeometric polynomials available in the literature. For example

(iii) If we put $z_1 = 1, A_{n,k} = \dfrac{\Gamma(1+\eta+\lambda n)}{n!\,\Gamma(1-\eta+\lambda k)}$ $\qquad \ldots (3.5)$

in (3.3) , replace γ and δ by γ/λ and δ/λ respectively, we get the following interesting integral :

$$\int_0^t (t-x)^{\rho} x^{\sigma} P_u^{(\alpha,\beta)}(1-z'x) \log(t-x)\, Z_n^m\,[(t-x)^{\gamma} x^{\delta};\lambda]$$

$$H_{P\,Q}^{M\,N}\left[z(t-x)^{\mu} x^{\nu} \,\middle|\, \begin{matrix}(a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,Q}\end{matrix}\right] dx = \sum_{k=0}^{[n/m]} \sum_{w=0}^{u} \sum_{h=0}^{M} \sum_{R=0}^{\infty}$$

$$t^{\rho+\sigma+1} \frac{(-n)_{mk}\,\Gamma(1+\eta+\lambda n)\,\Gamma(\alpha+u+1)\,(-u)_w\,(\alpha+\beta+u+1)_w}{k!\,n!\,\Gamma(1-\eta+\lambda k)\,\Gamma(\alpha+w+1)\,u!\,w!\,R!}$$

$$\frac{\Gamma(1+\rho+\frac{\gamma}{\lambda}k+\mu\,\xi_{h,R})\,\Gamma(1+\sigma+w+\frac{\delta}{\lambda}k+\nu\,\xi_{h,R})}{\Gamma(2+\rho+\sigma+w+(\frac{\gamma+\delta}{\lambda})k+(\mu+\nu)\,\xi_{h,R})}\Big[\log t +$$

$$\psi(1+\rho+\frac{\gamma}{\lambda}k+\mu\,\xi_{h,R}) - \psi\left(2+\rho+\sigma+w+\left(\frac{\gamma+\delta}{\lambda}\right)k+(\mu+\nu)\,\xi_{h,R}\right)\Big]$$

$$f(\xi_{h,R})\,t^{(\gamma+\delta)k}\,(\tfrac{1}{2}z't)^{w}\,(zt^{\mu+\nu})^{\xi_{h,R}} \qquad \ldots (3.6)$$

where $Z_n^m(x;\lambda)$ stands for the Kohnauser biorthogonal polynomial (Konhauser [5]).

(iv) If we take $u = 0$, $t = 1$ in (3.3) and replacing x by $\dfrac{1+y}{2}$ therein we easily get the integral recently given by Kant and Kaul ([4], p. 6, Eq. (15). The other integrals (16),(17) and (18) due to them are also special cases of our main integrals.

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MAGNETOHYDRODYNAMIC TEMPERATURE DISTRIBUTION OF TWO IMMISCIBLE VISCOUS LIQUIDS BETWEEN TWO PARALLEL PLATES

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ABSTRACT

This paper is concerned with unsteady flow of two conducting, incompressible,viscous, immiscible liquids between two parallel plates in the presence of uniform magnetic field applied perpendicular to the flow region. In the analysis, it has been assumed that the upper plate is moving with transient velocity while the lower plate is fixed. Velocity of the liquids and temperature at the plates have been obtained, following Soundalgekar et al. (1990). The effect on magnetic field, Reynolds number on the velocities of the lower and upper liquids and the temperature distribution of the lower and upper liquids for different values of Eckert number have been studied with the help of tables.

1. Introduction

Heat transfer in unsteady flows of immiscible viscous liquids in porous channel assumes importance due to its important applications in ground water hydrology, aero-dynamics and petroleum industry. However, this aspect has received little attention in the literature. Hartmann (1937) has done the pioneer work in the field of magnetohydrodynamics.

Temperature distribution in poiseuille flow between two parallel flat plates has been studied by Dube (1970). Mathur et al. (1972), Sacheti and Bhatt (1973) have considered the unsteady flow between two plates when lower plate is oscillating and upper plate is moving uniformely in their own planes in the presence and absence of a magnetic field respectively. Mathur and Jain (1981) have studied the magnetohydrodynamic temperature distribution for an unsteady laminar flow between two parallel plates.Sacheti et al. (1989) have discussed heat transfer in steady flow of immiscible fluids in a channel bounded below by a naturally permeable wall. Bhargava and Sacheti (1989) have investigated heat transfer in generalised Couette flow of

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two immiscible Newtonian fluids through a porous channel. Pal and Pal (1989) have investigated the transient temperature distribution in skin and subcutaneous tissues with improved variations in bio-physical parameters under various environmental conditions. Recently, Soundalgekar et al. (1990) have considered heat transfer in MHD unsteady stagnation point flow with variable wall temperature.

In the present paper, unsteady flow of two conducting, incompressible, viscous liquids between two parallel plates in presence of uniform magnetic field has been studied. Velocity of the liquids and temperature at the plates have been obtained, following Soundalgekar et al. (1990). The effect on magnetic field, Reynolds number on the velocities of the lower and upper liquids and the temperature distribution of the lower and upper liquids for different values of Eckert number have been studied with the help of tables.

2. Formulation of the Problem

Let us consider the laminar flow of two incompressible immiscible viscous, electrically, conducting liquids between two parallel plates. A constant magnetic field B_0 is applied perpendicular to the flow region. The depth betwen the plates is $2d$. The x-axis is assumed in the line flow along the central line which passes through the midway of the parallel plates and the y-axis is taken perpendicular to it. It is further assumed that the lower plate is fixed and the upper plate is moving with transient velocity U. The governing equations of motion and energy for the present configuration are :

$$\frac{\partial u_i}{\partial t} = \nu_i \frac{\partial^2 u_i}{\partial y^2} - \frac{\sigma_i}{\rho_i} B_0^2 u_i \qquad \ldots (2.1)$$

$$\frac{\partial T_i}{\partial t} = \frac{K}{\rho_i c_p} \frac{\partial^2 T_i}{\partial y^2} + \frac{\mu_i}{\rho_i c_p} \left(\frac{\partial u_i}{\partial y} \right)^2 \quad (i = 1, 2) \qquad \ldots (2.2)$$

where in these equations, ν_i is the kinematic coefficient of viscosity, u_i is the velocity of the liquids, σ_i is the electrical conductivity of the liquids, ρ_i is the density of the liquids, c_p is the specific heat at constant pressure, K is the thermal conductivity, t is the time μ_i is the coefficient of viscosity, B_0 is the magnitude of the magnetic field and the temperture is denoted by T_i.

The boundary conditions for the lower liquid are

$$\left.\begin{aligned} u_1 &= U_0, \quad T_1 = T_d \quad \text{at} \quad y = 0 \\ u_1 &= 0, \quad T_1 = T_0 \quad \text{at} \quad y = -d \end{aligned}\right\} \qquad \ldots (2.3)$$

The boundary conditions for the upper liquid are

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$$\left.\begin{aligned} u_2 &= U_0, \quad T_2 = T_d \quad \text{at} \quad y = 0 \\ u_2 &= U_0 U, \quad T_2 = T_0 \quad \text{at} \quad y = d \end{aligned}\right\} \qquad \ldots (2.4)$$

We introduce the following non-dimensional variables-

$$u_i^* = \frac{u_i}{U_0}, \quad y^* = \frac{y}{d}, \quad t^* = \frac{t}{d}, \quad T_i^* = \frac{T_i - T_0}{T_d - T_0}.$$

Using the above non-dimensional variables, the equations (2.1) and (2.2) after neglecting the asterisks over them are reduced to

$$\frac{\partial u_i}{\partial t} = \frac{1}{R_i} \frac{\partial^2 u_i}{\partial y^2} - M_i^2 u_i \qquad \ldots (2.5)$$

and

$$\frac{\partial T_i}{\partial t} = \frac{1}{R_i E_{c_i}} \frac{\partial^2 T_i}{\partial y^2} + \frac{P_r}{R_i} \left(\frac{\partial u_i}{\partial y}\right)^2 \qquad \ldots (2.6)$$

where,

$$\frac{\nu_i}{d} = \frac{1}{R_i} \text{ (Reynolds number)}, \quad \frac{K}{\mu_i c_p} = \frac{1}{E_{c_i}} \text{ (Eckert number).}$$

$$B_0 \left(\frac{\sigma_i^d}{\rho_i}\right)^{1/2} = M_i \text{ (Hartmann number)}, \quad \frac{U_0^2}{(T_d - T_0) c_p} = P_r \text{ (Prandtl number)}$$

The non-dimensional boundary conditions for lower liquid are

$$\left.\begin{aligned} u_1 &= 1, \quad T_1 = 1 \quad \text{at} \quad y = 0 \\ u_1 &= 0, \quad T_1 = 0 \quad \text{at} \quad y = -1 \end{aligned}\right\} \qquad \ldots (2.7)$$

The non-dimensional boundary conditions for the upper liquid are

$$\left.\begin{aligned} u_2 &= 1, \quad T_2 = 1 \quad \text{at} \quad y = 0 \\ u_2 &= U, \quad T_2 = 0 \quad \text{at} \quad y = 1 \end{aligned}\right\} \qquad \ldots (2.8)$$

3. Solution of the Problem

Following Soundalgekar et al. (1990), we assume

$$\left.\begin{aligned} u_i &= u_i^{(0)}(y) + \epsilon\, e^{-nt}\, u_i^{(1)}(y) \\ T_i &= T_i^{(0)}(y) + \epsilon\, e^{-nt}\, T_i^{(1)}(y) \end{aligned}\right\} \qquad \ldots (3.1)$$

Substituting the values of u_i and T_i in the equations (2.5) and (2.6), we obtain.

$$\frac{\partial^2 u_i^{(0)}}{\partial y^2} - M_i^2 R_i u_i^{(0)}(y) = 0 \qquad \ldots (3.2)$$

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$$\frac{\partial^2 u_i^{(1)}}{\partial y^2} - R_i (M_i^2 - n)\, u_i^{(1)}(y) = 0 \qquad \ldots (3.3)$$

$$\frac{\partial^2 T_i^{(0)}}{\partial y^2} + P_r E_{c_i} \left(\frac{\partial u_i^{(0)}}{\partial y} \right)^2 = 0 \qquad \ldots (3.4)$$

$$\frac{\partial^2 T_i^{(1)}}{\partial y^2} + nR_i E_{c_i}\, T_i^{(1)}(y) = -2P_r E_{c_i} \frac{\partial u_i^{(0)}}{\partial y} \cdot \frac{\partial u_i^{(1)}}{\partial y} \qquad \ldots (3.5)$$

Taking $i = 1$ and $i = 2$, the solutions of equations (3.2) and (3.3) under the boundary conditions (2.7) and (2.8), are

$$u_1^{(0)} = \frac{\sinh b_1 (1 + y)}{\sinh b_1} \qquad \ldots (3.6)$$

$$u_2^{(0)} = \frac{\sinh b_2 (1 - y) + U \sinh b_2 y}{\sinh b_2} \qquad \ldots (3.7)$$

$$u_1^{(1)} = u_2^{(1)} = 0 \qquad \ldots (3.8)$$

where $b_1 = M_1 \sqrt{R_1}$, $b_2 = M_2 \sqrt{R_2}$.

Taking $i = 1$ and $i = 2$, the solution of equation (3.4) and (3.5) with the help of equations (3.6) and (3.7) under the boundary conditions (2.7) and (2.8), are

$$T_1^{(0)} = 1 - \frac{P_r E_{c_1} b_1^2}{2 \sinh^2 b_1} \left[\frac{y^2}{2} + \frac{\cosh 2b_1 (1 + y)}{4b_1^2} \right] + \left[1 - \frac{P_r E_{c_1} b_1^2}{2 \sinh^2 b_1} \left\{ \frac{1}{2} + \frac{1}{4b_1^2} - \frac{\cosh 2b_1}{4b_1^2} \right\} \right] y + \frac{P_r E_{c_1} b_1^2 \cosh 2b_1}{8b_1^2 \sinh^2 b_1} \qquad \ldots (3.9)$$

$$T_2^{(0)} = 1 - \frac{P_r E_{c_2} b_2^2}{\sinh^2 b_2} \left[\frac{\cosh 2b_2 y}{4b_2^2} \left\{ \frac{U^2}{2} - U \cosh b_2 \right\} + \frac{y^2}{2} \left\{ \frac{U^2}{2} + \frac{1}{2} - U \cosh b_2 \right\} + \frac{\cosh 2b_2 (1 - y)}{8b_2^2} - \frac{U \sinh b_2 \sinh 2b_2 y}{4b_2^2} \right]$$

$$- y - \frac{b_2 P_r E_{c_2}}{\sinh^2 b_2} \left[\frac{\cosh 2b_2}{4b_2^2} \left\{ \left(\frac{1}{2} - \frac{U^2}{2} + U \cosh b_2 \right) y - \frac{1}{2} \right\} \right.$$

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$$+\frac{U}{4b_2^2}\left\{\left(\frac{U}{2}-\frac{1}{2U}-\cosh b_2\right)y-\left(\frac{U}{2}-\cosh b_2\right)\right\}$$

$$-\frac{y}{2}\left\{\frac{U^2}{2}+\frac{1}{2}-U\cosh b_2\right\}+y\,\frac{U\sinh 2b_2\sinh b_2}{4b_2^2}\Bigg] \qquad \ldots (3.10)$$

and $$T_1^{(1)}=T_2^{(1)}=0 \qquad \ldots (3.11)$$

On putting the values of $u_1^{(0)}, u_2^{(0)}, u_1^{(1)}, u_2^{(1)}, T_1^{(0)}, T_2^{(0)}, T_1^{(1)}$ and $T_2^{(2)}$ in equation (3.1), we obtain

$$u_1=\frac{\sinh b_1\,(1+y)}{\sinh b_1} \qquad \ldots (3.12)$$

$$u_2=\frac{\sinh b_2\,(1-y)+U\sinh b_2 y}{\sinh b_2} \qquad \ldots (3.13)$$

$$T_1=1-\frac{P_r E_{c_1} b_1^2}{2\sinh^2 b_1}\left[\frac{y^2}{2}+\frac{\cosh 2b_1\,(1+y)}{4b_1^2}\right]+$$

$$\left[1-\frac{P_r E_{c_1} b_1^2}{2\sinh^2 b_1}\left\{\frac{1}{2}+\frac{1}{4b_1^2}-\frac{\cosh 2b_1}{4b_1^2}\right\}\right]y+\frac{P_r E_{c_1} b_1^2\cosh 2b_1}{8b_1^2\sinh^2 b_1} \qquad \ldots (3.14)$$

and

$$T_2=1-\frac{P_r E_{c_2} b_2^2}{2\sinh^2 b_2}\left[\frac{\cosh 2b_2 y}{4b_2^2}\left\{\frac{U^2}{2}-u\cosh b_2\right\}\right.$$

$$+\frac{y^2}{2}\left\{\frac{U^2}{2}+\frac{1}{2}-U\cosh b_2\right\}+\frac{\cosh 2b_2\,(1-y)}{8b_2^2}$$

$$\left.-\frac{U\sinh b_2\sinh 2b_2 y}{4b_2^2}\right]-y-\frac{b_2^2 P_r E_{c_2}}{\sinh^2 b_2}\left[\frac{\cosh 2b_2}{4b_2^2}\right.$$

$$\left\{\left(\frac{1}{2}-\frac{U^2}{2}+U\cosh b_2\right)y-\frac{1}{2}\right\}+\frac{U}{4b_2^2}\left\{\left(\frac{U}{2}-\frac{1}{2U}-\cosh b_2\right)y\right.$$

$$\left.\left.-\left(\frac{U}{2}-\cosh b_2\right)\right\}-\frac{y}{2}\left\{\frac{U^2}{2}+\frac{1}{2}-U\cosh b_2\right\}\right.$$

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$$+ y \frac{U \sinh 2b_2 \sinh b_2}{4b_2^2} \Bigg] \qquad \ldots (3.15)$$

Table-I

Velocity of the lower liquid defined in equation (3.12) for different values of Hartmann number M_1.

y	$M_1 = 1.0$	$M_1 = 3.0$	$M_1 = 5.0$
0	1.00000	1.00000	1.00000
− 0.1	0.81112	0.54880	0.36788
− 0.2	0.65499	0.30118	0.13544
− 0.3	0.52506	0.16526	0.04979
− 0.4	0.41619	0.09065	0.01832
− 0.5	0.32403	0.04966	0.00674
− 0.6	0.24487	0.02709	0.00248
− 0.7	0.17554	0.01459	0.00091
− 0.8	0.11325	0.00748	0.00034
− 0.9	0.05551	0.00316	0.00011
− 1.0	0.00000	0.00000	0.00000

Table-II

Velocity of the lower liquid defined in equation (3.12) for different values of Reynolds number R_1.

y	$R_1 = 2.0$	$R_1 = 4.0$	$R_1 = 6.0$
0	1.00000	1.00000	1.00000
− 0.1	0.85030	0.81112	0.77903
− 0.2	0.71763	0.65499	0.60503
− 0.3	0.59934	0.52506	0.46752
− 0.4	0.49305	0.41619	0.35819
− 0.5	0.39664	0.32403	0.27048
− 0.6	0.30818	0.24487	0.19907
− 0.7	0.22589	0.17554	0.13966
− 0.8	0.14812	0.11325	0.08868
− 0.9	0.07333	0.05551	0.04304
− 1.0	0.00000	0.00000	0.00000

Table-III

Velocity of the upper liquid defined in equation (3.13) for different values of Hartmann number M_2 at $U = 2.0$

y	$M_2 = 1.5$	$M_2 = 3.5$	$M_2 = 5.5$

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0	1.00000	1.00000	1.00000
0.1	0.78015	0.47786	0.38528
0.2	0.63997	0.23152	0.14914
0.3	0.56513	0.11874	0.05955
0.4	0.54799	0.07449	0.02847
0.5	0.58681	0.07322	0.02537
0.6	0.68555	0.11427	0.04718
0.7	0.85491	0.22108	0.11535
0.8	1.11030	0.45555	0.29687
0.9	1.47960	0.95280	0.77009
1.0	2.00000	2.00000	2.00000

Table-IV

Velocity of the upper liquid defined in equation (3.13) for different values of Reynolds number R_2 at $U = 2.0$

y	$R_2 = 2.5$	$r_2 = 4.5$	$R_2 = 6.5$
0	1.00000	1.00000	1.00000
0.1	0.87478	0.78015	0.71608
0.2	0.79907	0.63997	0.53812
0.3	0.76844	0.56513	0.43988
0.4	0.78125	0.54799	0.40674
0.5	0.83828	0.58681	0.44384
0.6	0.94256	0.68555	0.52512
0.7	1.10020	0.85491	0.69415
0.8	1.32000	1.11030	0.96593
0.9	1.61430	1.47960	1.38080
1.0	2.00000	2.00000	2.00000

Table-V

Temperature distribution of the lower liquid defined in equation (3.14) for different values of Eckert number E_{c_1} at $P_r = 0.71$

y	$E_{c_1} = 0.10$	$E_{c_1} = 0.20$	$E_{c_1} = 0.30$
0.0	1.00000	1.00000	1.00000
− 0.1	0.90478	0.90956	0.91434
− 0.2	0.80744	0.81489	0.82234
− 0.3	0.70866	0.71733	0.72598
− 0.4	0.60886	0.61744	0.62659
− 0.5	0.50834	0.51673	0.52507
− 0.6	0.40730	0.41467	0.42197
− 0.7	0.30643	0.31183	0.31771
− 0.8	0.20415	0.20838	0.21253
− 0.9	0.10217	0.10442	0.10658
− 1.0	0.00004	0.00006	0.00008

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Table-VI

Temperature distribution of the upper liquid defined in equation (3.15) for different values of Eckert number E_{c_2} at $P_r = 0.71$ and $u = 2.0$

y	$E_{c_2} = 0.15$	$E_{c_2} = 0.25$	$E_{c_1} = 0.35$
0	0.99593	0.99320	0.99045
0.1	0.67094	0.51770	0.36484
0.2	0.34495	0.04126	– 0.26212
0.3	0.02376	– 0.42745	– 0.87821
0.4	– 0.28540	– 0.87623	– 1.46660
0.5	– 0.56901	– 1.28600	– 2.00020
0.6	– 0.81289	– 1.62230	– 2.43090
0.7	– 0.97224	– 1.82110	– 2.66920
0.8	– 0.97184	– 1.76220	– 2.54700
0.9	– 0.69042	– 1.21770	– 1.74450
1.0	– 1.14060	– 1.90170	– 2.66200

4. Discussion

The numerical values of the velocity of the lower liquid for different values of Hartmann number M_1 and Reynolds number R_1 with y have been listed in Table-I and Table II, respectively. A study of these two tables shows that the velocity of the lower liquid decreases as the intensity of magnetic field M_1 increases and similar in the case when Reynolds number R_1 increases. Besides, we observe that on increasing y the velocity of the lower liquid decreases in both the tables.

The numerical values of the velocity of the upper liquid for different values of Hartmann number M_2 and Reynolds number R_2 with y at $U = 2.0$ have been listed in Table-III and Table IV, respectively. It is observed from these tables that the velocity of the upper liquid decreases as the intensity of the magnetic field M_2 increases. Again when Reynolds number R_2 increases the velocity also decreases for given value of parameter. It is interesting to note that when y increases the velocity of the upper liquid decreases while it increases again as y increases from $y = 0.5$ to $y = 1.0$.

The numerical values for the temperature of the lower liquid for different values of Eckert number E_{c_1} at $P_r = .71$ are shown in Table-V. We observe that the temperature of the lower liquid increases as the Eckert number E_{c_1} increases.Besides on increasing y temperature of the lower liquid decreases.

The numerical values for the temperature of the upper liquid for different values of the Eckert number E_{c_2} at $P_r = 0.71$ and $U = 2.0$ are

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shown in Table-VI. Here we observe that the temperature of the upper liquid decreases as the Eckert number E_{c_2} increases.Besides on increasing y temperature of the upper liquid decreases.

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SOME WEAKER FORMS OF FUZZY CONTINUOUS MAPPINGS

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ABSTRACT

Some weaker forms of fuzzy continuous mappings have been introduced and studied. Results pertaining to some of their preservation properties are also discussed.

1. INTRODUCTION

Using semi open sets in topological spaces, several weaker forms of continuity have been studied in the literature. For example, Bhamini [2] defined a mapping $f: X \to Y$ to be almost semi continuous if the inverse image of regularly open sets in Y are semi open in X. She [2] calls f to be weakly semi continuous if for every open set U of Y, $f^{-1}(U) \subset S - \text{int}\, f^{-1}(clU)$. Here S-int denotes the semi interior operator. Singal and Yadav [7] call f to be slightly semicontinuous if $f^{-1}(U)$ is semi open in X for every clopen subset U of Y. Here we study these mappings in the fuzzy setting. However, we use a more suggestive terminology for these mappings. We call a mapping $f: X \to Y$ to be $I(\alpha, \beta)$ if the inverse image of a set of type β is a set of type α. For example, a mapping $f: X \to Y$ is $I(o, clo)$ if the inverse image of each clopen set in Y is open in X. Bhamini's almost semi continuous mapping in our terminology is I (so, ro) and slightly semi continuous mapping of Singal and Yadav is I (so, clo). Here we study some of these mapping in the fuzzy setting.

2. PRELIMINARIES

Throughout this paper X and Y mean fuzzy topological spaces. I denotes the closed unit interval. The definitions of fuzzy sets, fuzzy topological spaces and other concepts about functions can be found in[3,8,9].

Let U be a fuzzy set of X. U is said to be fuzzy semi open set of X if there exists a fuzzy open set V such that $V \leq U \leq \text{cl}V$. Complement of a fuzzy semi open set is called fuzzy semi closed. S-int U (Semi interior of U) is defined as the suprimum of all fuzzy semi-open sets contained in U and S-cl$U$ (semi closure of U) as the infimum of all fuzzy semi closed sets containing U.

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A fuzzy set U of a fuzzy space X is called a fuzzy regular open set of X if int cl$U$ = U, and a fuzzy regular closed set of X if *cl int* U = U.

A fuzzy set U of X is called fuzzy clopen (resp. fuzzy semi clopen) set of X if U is fuzzy open and fuzzy closed (resp. fuzzy semi open and fuzzy semi closed). A fuzzy set U of X is said to be fuzzy δ^* – open if U can be written in the form $U = V U_i$, where U_i are fuzzy clopen sets.Complement of a fuzzy δ^* – openset is called fuzzy δ^* closed.

A mapping $f: X \to Y$ from a fuzzy space X to fuzzy space is called a, (i) **fuzzy open mapping** [1], if $f(A)$ is fuzzy open set of Y, for each fuzzy open set A of X; (ii) **fuzzy closed mapping** [1], if $f(A)$ is fuzzy closed set of Y, for each fuzzy closed set A of X; (iii) **fuzzy presemi open mapping**, if $f(A)$ is fuzzy semiopen set of Y for each fuzzy semi open A of X.

A fuzzy space X is said to be (i) fuzzy semi compact (**resp. fuzzy slightly compat**) if every fuzzy semi open (resp. fuzzy clopen) cover of X has a finite subcover; (ii) fuzzy **strongly S-closed** if every fuzzy semi open cover of X has a finite subfamily such that semi closures of whose members cover X.

A fuzzy space X is said to be (i) **fuzzy semi T_0 ,** if every fuzzy set A of X can be written in the form $A = \underset{i \in I}{\vee} \underset{j \in J_i}{\wedge} U_{ij}$, where U_{ij} are fuzzy semi open or fuzzy semi closed sets; (ii) **fuzzy semi T_1**, if every fuzzy set A of X can be written in the form $A = \underset{i \in I}{\vee} U_i$, where U_i are fuzzy semi closed sets; (iii) **fuzzy semi T_2**, if every fuzzy set A of X can be written in the form $A = \underset{i \in I}{\vee} \underset{j \in J_i}{\wedge} U_{ij} = \underset{i \in I}{\vee} \underset{j \in J_i}{\wedge} S - \text{cl} U_{ij}$, where U_{ij} are fuzzy semi open sets, (iv) fuzzy S-regular, if every fuzzy open set U can be written in the form $U = V_{i \in I} U_i$, where U_i are fuzzy semi open sets with $S - \text{cl}\, U_i \leq U$; ($v$) fuzzy S-normal, if for any fuzzy closes set K and fuzzy open set U such that $K \leq U$, there exists a fuzzy V such that

$$K \leq S - \text{int}\, V \leq S - clV \leq U.$$

A fuzzy space (X, τ) is called a fuzzy semi regular space [1] if the collection of all fuzzy regular open sets of X forms a base for fuzzy topology.

A fuzzy space X is called **product related** [1] to a fuzzy space Y if for any fuzzy set U of X and V of Y whenever $A' < U$ and $B' < V$ implies $(A' \times 1) \vee (1 \times B') \geq U \times V$, where A is fuzzy open set of X and B is fuzzy open set of Y, there exists a fuzzy open set A_1 of X and a fuzzy

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open set B_1 of Y such that $A_1' \geq U$ or $B_1' \geq V$ and $(A_1' \times 1) \vee (1 \times B_1')$ $= (A' \times 1) \vee (1 \times B')$.

3. FUZZY $I(so, ro)$ MAPPINGS

3.1 Definition : A mapping $f : X \longrightarrow Y$ from a fuzzy space X to a fuzzy space Y is said to be fuzzy $I(so, ro)$ if inverse image of every fuzzy regular open set of Y is fuzzy semi open set of X.

3.2 Remark : Fuzzy $I(so, ro)$ mappings are a generalization of the concept of almost semi continuous mapping studied by Bhamini [2].

3.3 Theorem : For a mapping $f : X \to Y$ from a fuzzy space X to a fuzzy space Y, the following are equivalent : (a) f is fuzzy $I(so, ro)$; (b) the inverse image of every fuzzy closed set of Y is fuzzy semi closed set of X; (c) $f^{-1}(U)) \leq S - \text{int}\,(f^{-1}\,(\text{int cl}U))$ for every fuzzy open set U of Y; (d) $S - cl\,(f^{-1}\,(cl\,\text{int}\,V)) \leq f^{-1}\,(V)$ for every fuzzy closed set V of Y.

Proof : $(a) \Leftrightarrow (b)$, because for any fuzzy set U of Y, we have $f^{-1}(U') = f^{-1}(U))'$. $(a) \Rightarrow (c)$. Let U be a fuzzy open set of Y. Then $U \leq \text{int}\,clU$ and hence $f^{-1}(U) \leq f^{-1}\,(\text{int}\,clU)$. Since int clU is a fuzzy regular open set of Y, $f^{-1}(\text{int}\,clU)$ is a fuzzy semi open set of X. Thus $f^{-1}(U) \leq f^{-1}\,(\text{int}\,clU) = S - \text{int}\,f^{-1}\,(\text{int}\,clU)$. $(c) \Leftrightarrow (d)$. Taking complement of both sides of (c) we get (d), and vice versa. $(d) \Rightarrow (b)$. Let U be a fuzzy regular closed set of Y. By (d) we have

$$S - cl\,(f^{-1}\,(cl\ \text{int}\ U)) = S - cl\,f^{-1}(u) \leq f^{-1}(U).$$

Thus $S - cl\,f^{-1}(U) = f^{-1}(U)$, which is fuzzy semi closed.

3.4 theorem. If $f : X \to Y$ be a fuzzy $I(so, ro)$ mapping, then for every fuzzy open set U of Y, $S - cl\,f^{-1}(U) \leq f^{-1}(clU)$

Proof : Let $f : X \to Y$ be a fuzzy $I(so, ro)$ mapping from a fuzzy space X to a fuzzy space Y. Then $f^{-1}(U) \leq f^{-1}\,(clU)$ for every fuzzy set U of Y. If U is fuzzy open, then clU is fuzzy regular closed. Hence $f^{-1}(clU)$ is fuzzy semi closed set of X. Thus we have $S - cl\,f^{-1}\,(U) \leq f^{-1}\,(clU)$.

Clearly, every fuzzy $I(0, ro)$ mapping as well as every fuzzy $I(so, o)$ mapping is fuzzy $I(so, ro)$. But the converse need not be true as is shown by the following examples :

3.5 Example : Let U_1, U_2 and U_3 be fuzzy sets of I defined as follows :

for each $x \in I$,

$$U_1(x) = x, \qquad 0 \leq x \leq 1;$$

$$U_2\,(x) = 1 - x, \qquad 0 \leq x \leq 1;$$

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$$U_3(x) = x, \qquad 0 \le x \tfrac{1}{2};$$
$$= 0, \qquad \tfrac{1}{2} < x \le 1.$$

Consider fuzzy topologies $\tau_1 = \{0, U_1 \wedge U_2, 1\}$ and $\tau_2 = \{0, U_1, U_2, U_1 \wedge U_2, U_1 \vee U_2, 1\}$ on I and the mapping $f: (I, \tau_1) \to (I, \tau_2)$ defined by $f(x) = x$, for each $x \in I$. It is clear that $U_1, U_2, U_1 \vee U_2$ and $U_1 \wedge U_2$ being fuzzy open and fuzzy closed are fuzzy regular open sets in (I, τ_2). It is obvious that inverse image of every fuzzy regular open set in (I, τ_2) is fuzzy semi open in (I, τ_1), because in (I, τ_1), $cl\,(U_1 \wedge U_2) = 1$ and $U_1 \wedge U_2 \le f^{-1}(U_1) \le 1$, $U_1 \wedge U_2 \le f^{-1}(U_2) \le 1$, $U_1 \wedge U_2 \le f^{-1}(U_1 \wedge U_2) \le 1$. Hence the mapping is fuzzy $I(so, ro)$. The mapping is not fuzzy $I(o, ro)$, because none of $f^{-1}(U_1), f^{-1}(U_2)$ and $f^{-1}(U_1 \vee U_2)$ is open in (I, τ_1).

3.6 Example : We refer to fuzzy sets as defined in Example 3.5 Consider the fuzzy topologies $\tau_1 = \{0, U_1 \wedge U_2, 1\}$ and $\tau_2 = \{0, U_1, U_2, U_3, U_1 \vee U_2, U_1 \wedge U_2, 1\}$ on I and the mapping $f: (I, \tau_1) \to (I, \tau_2)$ defined by $f(x) = x$ for every $x \in I$ It is clear that the mapping is fuzzy $I(so, ro)$. Also, because O is the only fuzzy open set contained in $f^{-1}(U_3), f^{-1}(U_3) = U_3$ is not fuzzy semi open set of (I, τ_1). Hence f is not a fuzzy $I(so, o)$ mapping.

3.7 Remark : Fuzzy $I(o, ro)$ and fuzzy $I(so, o)$ mappings have been studied under the names fuzzy almost continuous and fuzzy semi continuous [1].

3.8 Theorem : If $f: X \to Y$ is a fuzzy $I(so, ro)$ Mapping from a fuzzy space X to fuzzy semi regular space Y. Then f is fuzzy $I(so, o)$.

Proof : Let U be a fuzzy open set of Y. Then $U = \underset{i \in I}{\vee} U_i$, where U_i are fuzzy regular open sets of Y; since Y is fuzzy semi regular. Now, using Theorem 3.3 (c), we get

$$f^{-1}(U) = f^{-1}\left(\underset{i \in I}{\vee} U_i\right) = \underset{i \in I}{\vee} f^{-1}(U_i)$$
$$\le \underset{i \in I}{\vee} S - \operatorname{int} cl U_i) = \underset{i \in I}{\vee} S - \operatorname{int} f^{-1}(U_i)$$
$$\le S - \operatorname{int}\left(\underset{i \in I}{\vee} f^{-1}(U_i)\right) = S - \operatorname{int} f^{-1}(U),$$

which shows that $f^{-1}(U)$ is fuzzy semi open set of X. Hence f is fuzzy $I(so, o)$.

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3.9 Theorem : Let $f: X \to Y$ be a fuzzy $I(so, so)$ and fuzzy pre semi open mapping and let $g: Y \to Z$ be any mapping. Then $gof: X \to Z$ is fuzzy $I(so, ro)$ if g is fuzzy $I(so, ro)$.

Proof : The if part is obvious. To prove the only if part, let $gof: X \to Z$ be a fuzzy I (so, ro) mapping. Let U be a fuzzy regular open set of Z. then $(gof)^{-1}(U) = f^{-1}(g^{-1}(U))$ is fuzzy semi open set of X and hence $f(f^{-1}(g^{-1}(U)) = g^{-1}(U)$ is fuzzy semi open set of Y. Therefore g is fuzzy $I(so, ro)$.

3.10 Remark : Fuzzy $I(so, so)$ mappings have been studied under the name fuzzy irresolute by Mukherjee and Sinha [6].

3.11 Theorem : Let $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ be fuzzy $I(so, ro)$ mappings. If Y_1 is product related to Y_2, then the product mapping $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is fussy $I(so, ro)$.

Proof: Let X_1, X_2, Y_1 and Y_2 be fuzzy spaces such that Y_1 is product related to Y_2. Let $U \equiv \vee (U_i \times U_j)$ be a fuzzy open set of $Y_1 \times Y_2$, where U_1 and U_j are open sets of Y_1 and Y_2 respectively. Since f_1 and f_2 are fuzzy $I(so, ro)$ mappings, we have

$$(f_1 \times f_2)^{-1}(U) = \vee \{f_1^{-1}(U_i) \times f_2^{-1}(U_j)\}$$

$$\leq \vee \{S - \text{int}\, f_1^{-1}(\text{int}\, cl\, U_i) \times S - \text{int}\, f_2^{-1}(\text{int}\, cl\, U_j)\}$$

$$\leq \vee \{S - \text{int}\, (f_1^{-1}(\text{int}\, cl\, U_i) \times f_2^{-1}(\text{int}\, cl\, U_j))\}$$

$$\leq S - \text{int}\, \{\vee (f_1 \times f_2)^{-1}(\text{int}\ cl\, U_i \times \text{int}\, cl\, U_j)\}$$

$$= S - \text{int}\ \{\vee f_1 \times f_2)^{-1}(\text{int}\, cl\, (U_i \times U_j))\}$$

$$\leq S - \text{int}\ \{(f_1 \times f_2)^{-1}(\text{int}\ cl\, (\vee (U_i \times U_j)))\}$$

$$= S\, \text{int}\, (f_1 \times f_2)^{-1}(\text{int}\, clU).$$

Thus by theorem 3.3 (c), $f_1 \times f_2$ is fuzzy $I(so, ro)$.

3.12 Theorem : Let X and Y be fuzzy spaces such that X is product related to Y and let $f: X \to Y$ be a mapping. If the graph mapping $g: X \to X \times Y$ of f is fuzzy $I(so, ro)$, then f is also fuzzy $I(so, ro)$.

Proof : Suppose that $g: X \to X \times Y$ is fuzzy $I(so, ro)$ mapping and U is a fuzzy open set of Y. We have

$$f^{-1}(U) = 1 \wedge f^{-1}(U) = g^{-1}(1 \times U)$$

$$\leq S - \text{int}\, g^{-1}(\text{int}\, cl\, (1 \times U))$$

$$= S - \text{int}\, g^{-1}(\text{int}\, (1 \times clU))$$

$$= S - \text{int}\, g^{-1}(1 \times \text{int}\, clU)$$

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$= S - \text{int}\, f^{-1}(\text{int}\, clU)$.

Hence by Theorem 3.3 (c), f is fuzzy I (*so, ro*).

4. FUZZY WEAKLY SEMI CONTINUOUS MAPPINGS

4.1 Definition : A mapping $f: X \to Y$ from a fuzzy space X to a fuzzy space Y is said to be fuzzy weakly semi continuous if for every fuzzy open set U of Y,

$$f^{-1}(U) \leq S - \text{int}\, f^{-1}(clU).$$

4.2 Remark : Fuzzy weakly semi continuous mappings are the genralization of weakly semi continuous mappings intorduced by Bhamini [2].

4.3 Definition [1] : A mapping $f: X \to Y$ from a fuzzy space X to a fuzzy space Y is called fuzzy weakly continuous mapping if for each fuzzy open set U of Y, $f^{-1}(U) \leq \text{int}\, f^{-1}(clU)$.

Clearly, every fuzzy weakly continuous as well as every fuzzy I(*so, ro*) mapping is fuzzy weakly semi continuous. For the converse, we have

4.4 Example : Let $X = \{x, y, z\}$ and U_1, U_2 be fuzzy sets of X defined as follows :

$U_1(x) = 0.2, U_1(y) = 0.4, U_1(z) = 0.3;$

$U_2(x) = 0.4, U_2(y) = 0.5, U_2(z) = 0.3;$

Let $\tau_1 = \{0, U_2, 1\}$ and $\tau_2 = \{0, U_1, 1\}$. Let $f: (X, \tau_1) \to (X, \tau_2)$ be the identity, mapping. By easy computations it is clear that in (X, τ_2), $cl\, U_1 = U_1'$, int $clU_1 = U_1$ and in (X, τ_1), $S - \text{int}\, f^{-1}(U_1') = U_2'$ and $S - \text{int}\, f^{-1}(U_1) = 0$. Hence $f^{-1}(U_1) \leq S - \text{int}\, f^{-1}(clU_1)$, which shows that mapping is fuzzy weakly semi continuous. However, $f^{-1}(U_1) > \text{int}\, f^{-1}(\text{int}\, clU_1)$showing that the mapping is not fuzzy I(*so, ro*).

4.5 Example : We take fuzzy sets U_1, U_2 and fuzzy topologies as defined in previous example, and $f: (X, \tau_2) \to (X, \tau_1)$ the identity mapping. It is clear that in (X, τ_1), $clU_2 = U_2'$, and in (x, τ_2), $\text{int}\, f^{-1}(U_2') = U_1$ and $S - \text{int}\, f^{-1}(U_2') = U_2'$. Hence $f^{-1}(U_2) \leq S - \text{int}\, f^{-1}(U_2')$ showing that the mapping is fuzzy weakly semi continuous. However, $f^{-1}(U_2) > \text{int}\, f^{-1}(U_2')$, hence the mapping is not fuzzy weakly continuous.

4.6 Theorem : Let $f: (X, \tau_1) \to (X, \tau_2)$ be a fuzzy weakly semi continuous mapping from a fuzzy space X to a fuzzy regular space Y. Then f is fuzzy I(*so, o*).

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Proof : Let U be a fuzzy open set of Y. Since Y is fuzzy regular space $U = \underset{i \in I}{\vee} U_i$, where U_i are fuzzy open sets of Y with $cl\ U_i \leq U$. Now, since f is fuzzy weakly semi continuous, we have.

$$f^{-1}(U) = f^{-1}(\underset{i \in I}{\vee} U_i) = \underset{i \in I}{\vee} f^{-1}(U_i)$$

$$\leq \underset{i \in I}{\vee} S - \text{int}\, f^{-1}(cl\ U_i)$$

$$\leq S - \text{int}\,(\underset{i \in I}{\vee} f^{-1}(cl U_i)) = S - \text{int}\, f^{-1}(U),$$

which shows that $f^{-1}(U)$ is a fuzzy semi open set of X. Thus, f is fuzzy $I(so, o)$.

4.7 Theorem : Let X_1, X_2, Y_1 and Y_2 be fuzzy spaces such that Y_1 is product related to Y_2. Then, the product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of fuzzy weakly semi continuous mapping $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is fuzzy weakly semi continuous.

Proof : Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be fuzzy weakly semi continuous mappings such that fuzzy space Y_1 is product related to Y_2. Let $U \equiv \vee (U_i \times U_j)$ be a fuzzy open set of $Y_1 \times Y_2$, where U_i and U_j are fuzzy open sets of Y_1 and Y_2 respectively. We have, by fuzzy weakly semi continuity of f_1 and f_2

$$(f_1 \times f_2)^{-1}(U) = \vee \{f_1^{-1}(U_i) \times f_2^{-1}(U_j)\}$$

$$\leq \vee \{S - \text{int}\, f_1^{-1}(cl U_i) \times S - \text{int}\, f_2^{-1}(cl U_j)\}$$

$$\leq \vee \{S - \text{int}\,(f_1^{-1}(cl\ U_i) \times f_2^{-1}(cl U_j))\}$$

$$\leq S - \text{int}\,\{\vee f_1 \times f_2)^{-1}(cl\ U_i \times cl\ U_j)\}$$

$$= S - \text{int}\,\{\vee (f_1 \times f_2)^{-1}(cl\ (U_i \times U_j))\}$$

$$= S - \text{int}\,(f_1 \times f_2)^{-1}(clU).$$

Thus by the definition of fuzzy weakly semi continuity, $f_1 \times f_2$ is fuzzy weakly semi continuous.

4.8 Theorem : Let X and Y be product related fuzzy spaces and $f : X \to Y$ be a mapping.

If the graph function $g : X \to X \times Y$ of f is fuzzy weakly semi continuous, then f is also fuzzy weakly semi continuous.

Proof : Let X and Y be fuzzy spaces such that X is product related to Y and let $f : X \to Y$ be a mapping. Suppose that the graph

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function $g : X \to X \times Y$ of f is fuzzy weakly semi continuous. Then for any fuzzy open set U of Y, we have

$$f^{-1}(U) = 1 \wedge f^{-1}(U) = g^{-1}(1 \times U) \leq S - \text{int}\, g^{-1}(cl\,(1 \times U)$$
$$= S - \text{int}\, g^{-1}(1 \times clU) = S - \text{int}\, f^{-1}(clU).$$

Thus by the definition, f is fuzzy weakly semi continuous.

5. FUZZY $I(so, clo)$ MAPPINGS

Slightly semi continuous mappings in general topology have been introduced by Singal and Yadav [7]. Here we generalize this notion to the fuzzy setting.

5.1 Definition : A mapping $f : X \to Y$ from a fuzzy topological space X to a fuzzy space Y is said to be fuzzy $I(so, clo)$ if inverse image of every fuzzy clopen set of Y is fuzzy semi open set of X.

5.2 Theorem : For a mapping $f : X \to Y$ from a fuzzy space X to a fuzzy spce Y, the following are equivalent :

(a) f is fuzzy $I(so, clo)$;

(b) Inverse image of every fuzzy clopen set of Y is fuzzy semi closed set of X.

(c) Inverse image of every fuzzy clopen set of Y is fuzzy semi clopen set of X.

(d) Inverse image of every fuzzy δ^*-open set of Y is fuzzy semi open set of X.

(e) Inverse image of every fuzzy δ^*-closed set of Y is fuzzy semi closed set of X.

Proof : It is obvious.

The following implications hold.

Fuzzy $I(0, clo) \Rightarrow$ Fuzzy $I(so, clo) \Rightarrow$ Fuzzy weakly semi continuity

We now show that none of the above implications is reversible.

5.2 Example : Let U_1 and U_2 be fuzzy sets of I defined as follows : for each $x \in I$,

$$U_1(x) = 2x, \qquad 0 \leq x \leq 1/2$$
$$= 1, \qquad 1/2 \leq x < 1;$$
$$U_2(x) = 2x, \qquad 0 \leq x \leq 1/4$$
$$= 1 - 2x, \qquad 1/4 \leq x < 1/2,$$
$$= 0, \qquad 1/2 \leq x \leq 1.$$

Consider fuzzy topologies $\tau_1 = \{0, U_2, 1\}$ and $\tau_2 = \{0, U_1, U_1', U_1 \vee U_1', U_1 \wedge U_1', 1\}$ on I and the mapping $f : (I, \tau_1) \to (I, \tau_2)$ defined by $f(x) = x$, for every $x \in I$. It is clear that $U_1, U_1', U_1 \vee U_1'$ and $U_1 \wedge U_1'$ are fuzzy clopen sets in (I, τ_2). In (I, τ_1)

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we have $clU_2 = U'_2$, $U_2 \leq f^{-1}(U_1) \leq U_2'$, $U_2 \leq f^{-1}(U_1') \leq U_2'$, $f^{-1}(U_1 \vee U_1') = U_2'$ and $f^{-1}(U_1 \wedge U_1') = U_2$. Thus the inverse image of every fuzzy clopen set in (I, τ_2) is fuzzy semi open in (I, τ_1). Hence f is fuzzy $I(so, clo)$. However, the inverse image of fuzzy clopen set U_1 in (I, τ_2) is not fuzzy open in (I, τ_1). Hence the mapping is not fuzzy $I(o, clo)$.

5.3 Remark : $I(o\, clo)$ maps in general topology have been studied under the name slightly continuous maps [5].

5.4 Example : Let $X = \{x, y, z\}$, and U_1, U_2, V_1, V_2 and V_3 be fuzzy sets of X defined as follows :

$U_1(x) = 0.1$, $U_1(y) = 0.4$, $U_1(z) = 0.2$;

$U_2(x) = 0.3$, $U_2(y) = 0.3$, $U_2(z) = 0.3$;

$V_1(x) = 0.4$, $V_1(y) = 0.5$, $V_1(z) = 0.3$;

$V_2(x) = 0.6$, $V_2(y) = 0.5$, $V_2(z) = 0.7$;

$V_3(x) = 0.8$, $V_3(y) = 0.6$, $V_3(z) = 0.8$;

Let $\tau_1 = \{0, U_2, 1\}$ and $\tau_2 = \{0, U_1, V_1, V_2, V_3, 1\}$.

If we define $f(X, \tau_1) \to (X, \tau_2)$ to be identity mapping, then it is easy to see that the inverse image of fuzzy clopen sets V_1 and V_2 in (I, τ_2) is fuzzy semi open in (I, τ_1). Thus the mapping f is fuzzy $I(so, clo)$. It is clear that in (I, τ_2), $cl\, U_1 = V_3'$ and in (I, τ_1), $S - \text{int}\, f^{-1}(V_3') = 0$. Hence U_1 is a fuzzy open set in (I, τ_2) with $f^{-1}(U_1) > S - \text{int}\, f^{-1}(clU_1)$. This shows that the mapping f is not fuzzy weakly semi continuous.

5.5 Theorem : If the graph function of a function is fuzzy $I(so,lo)$, then the function itself is fuzzy $I(so, clo)$.

Proof : Let $f : X \to Y$ be a mapping from a fuzzy space X to a fuzzy space Y such that the graph function $g : X \to X \times Y$ is fuzzy $I(so, clo)$. Let U be a fuzzy clopen set of Y. Then $1 \times U$ is a fuzzy clopen set of $X \times Y$. For each $x \in X$ we have.

$$g^{-1}(1 \times U)(x) = (1 \times U)\, g(x) = (1 \times U)(x, f(x))$$

$$= \min(1(x), U(f(x))) = 1 \wedge f^{-1}(U)(x) = f^{-1}(U)(x),$$

which is fuzzy semi open, because g is fuzzy $I(so, clo)$. Hence f is also fuzzy $I(so, clo)$.

5.6 Remark : Since the intersection of two fuzzy clopen sets is fuzzy clopen, the family of all fuzzy clopen sets of a fuzzy topological space (X, τ) forms a base for a fuzzy topology τ^* on X.

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5.7 Definition : Fuzzy topology τ^* generated by the set of all fuzzy clopen sets of (X, τ) is called the fuzzy 0- dimensionalization of τ. A fuzzy topological space (X, τ) is fuzzy 0-dimensional if $\tau^* = \tau$.

5.8 Theorem : A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy $I(so, clo)$ if $f : (X, \tau_1) \to (Y, \tau_2^*)$ is fuzzy $I(so, o)$.

The proof is obvious from Theorem 5.2 (e).

5.9 Corollary : If $f : X \to Y$ is a fuzzy $I(so, clo)$ mapping and Y is a fuzzy 0-dimensional space,then f is fuzzy $I(so, o)$.

5.10 Theorem : If $f : X \to Y$ is a fuzzy $I(so, so)$ fuzzy presemi open mapping from a fuzzy space X on to fuzzy space Y and $g : Y \to Z$ is any mapping, then gof is fuzzy $I(so, clo)$ iff g is fuzzy $I(so, clo)$.

Proof : Suppose that $gof : X \to Z$ is fuzzy $I(so, clo)$ mapping. Let U be a fuzzy clopen set of Z. Then $(gof)^{-1}(U) = f^{-1}(g^{-1}(U))$ is fuzzy semi open set of X. Since f is fuzzy pre semi open and surjective, hence $f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$is fuzzy semi open set of Y. Thus g is fuzzy $I(so, clo)$.

The converse is obvious.

5.11 Theorem : A fuzzy $I(so, clo)$ image of a fuzzy strongly S-closed space is fuzzy slightly compact.

Proof : Let $f : X \to Y$ be a fuzzy $I(so, clo)$ mapping from a fuzzy strongly S-closed space X on to a fuzzy space Y. Let $\{U_i\}_{i \in I}$ be a fuzzy clopen cover of Y. Since f is fuzzy $I(so,clo)$ $\{f^{-1}(U_i)\}$ is a fuzzy semi clopen cover of X. Since the space is fuzzy strongly S-closed, there exists a finite subset K of I such that

$$\bigvee_{i \in K} f^{-1}(U_i) = 1_X.$$

From the surjectivity of f, we have

$$f(\bigvee_{i \in K} f^{-1}(U_i)) = \bigvee_{i \in K} f(f^{-1}(U_i)) = \bigvee_{i \in K} U_i = 1_Y.$$

Thus Y is fuzzy slightly compact.

Since fuzzy semi compact space is fuzzy strongly S-closed, the following corollary is immediate :

5.12 Corollary : A fuzzy $I(so, clo)$ image of a fuzzy semi compact space is fuzzy slightly compact.

5.13 Theorem : Let $f : X \to Y$ be fuzzy $I(so, clo)$ injective mapping from a fuzzy space X to a fuzzy 0-dimensional space Y. Then X is fuzzy semi P whenever Y is fuzzy P, where $P \in \{T_0, T_1, T_2\}$ [4].

Proof : Since Y is fuzzy O-dimensional and f is fuzzy $I(so, clo)$ then by Corollary 5.9, f is fuzzy $I(so, o)$.

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Suppose Y is fuzzy T_0. Let A be a fuzzy set of X. Then $f(A) = \underset{i \in I}{\vee} \underset{j=\in J_i}{\wedge} U_{ij}$, where U_{ij} are fuzzy open or fuzzy closed sets of Y. Since f is fuzzy $I(so, o)$ and injective, we have

$$A = f^{-1}(f(A)) = f^{-1}(\underset{i \in I}{\vee} \underset{j \in J_i}{\wedge} U_{ij}) = \underset{i \in I}{\vee} \underset{j \in J_i}{\wedge} f^{-1}(U_{ij}),$$

where $f^{-1}(U_{ij})$ are fuzzy semi open or fuzzy semi closed sets of X. Thus X is fuzzy semi T_0.

Next, let Y be fuzzy T_1. If A is a fuzzy set of X, then $f(A)$ can be written in the form $f(A) = \underset{i \in I}{\vee} U_i$, where U_i are fuzzy closed sets of Y. Since f is fuzzy $I(so, o)$ injection.

We have

$$A = f^{-1}(f(A))) = f^{-1}(\underset{i \in I}{\vee} U_i) = \underset{i \in I}{\vee} f^{-1}(U_i),$$

where $f^{-1}(U_i)$ are fuzzy semi closed sets of X. Thus X is fuzzy semi T_1.

Finally, suppose that Y is fuzzy T_2 space. Let A be a fuzzy set of X. Then $f(A)$ being fuzzy set of Y can be written as

$f(A) = \underset{i \in I}{\vee} \underset{j \in J_i}{\wedge} U_{ij} = \underset{i \in I}{\vee} \underset{j \in J_i}{\wedge} cl\, U_{ij}$, where U_{ij} are fuzzy semi open sets of Y.

Since f is injective and $I(so, 0)$, we have

$$A = f^{-1}(f(A)) = f^{-1}(\underset{i \in I}{\vee} \underset{J \in J_i}{\wedge} U_{ij}) = f^{-1}\left(\underset{i \in I}{\vee} \underset{J \in J_i}{\wedge} cl\ U_{ij}\right)$$

$$= \underset{i \in I j \in J_i}{\vee \wedge} f^{-1}(cl\, U_{ij}) \geq \underset{i \in I}{\vee} \underset{J \in J_i}{\wedge} S - cl\, f^{-1}(U_{ij})$$

where $f^{-1}(U_{ij})$ are fuzzy open sets of X

Hence $A = A = \underset{i \in I j \in J}{\vee \wedge} f^{-1}(U_{ij}) = \underset{i \in I}{\vee} \underset{j \in J_i}{\wedge} S - cl\, f^{-1}(U_{ij})$.

Thus X is fuzzy semi T_2.

5.14 Theorem : Let $f : X \to Y$ be a fuzzy $I(so, clo)$ injection from a fuzzy space X to fuzzy 0-dimensional space Y.

(a) If Y is fuzzy regular [4] and f is fuzzy open (or fuzzy closed), then X is fuzzy S-regular.

(b) If Y is fuzzy normal [4] and f is fuzzy closed, then X is fuzzy S-normal.

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Proof : (a) Let A be a fuzzy open set of X. Since f is fuzzy open mapping , $f(A)$ is fuzzy open set of Y. Since Y is fuzzy regular, we have $f(A) = \underset{i \in I}{\vee} U_i$, where U_i are fuzzy open sets of Y with $cl\ U_i \leq f(A)$. By Corollary 5.9, f is fuzzy $I(so, o)$ and injective. We therefore have

$$A = f^{-1}(f(A)) = f^{-1}(\underset{i \in I}{\vee} U_i) = \underset{i \in I}{\vee} f^{-1}(U_i),$$

where $f^{-1}(U_i)$ are fuzzy semi open sets of X with $S = cl\, f^{-1}(U_i) \leq f^{-1}(cl\ U_i) \leq A$. Thus X is fuzzy S-regular.

(b) Let K and U be, respectively fuzzy closed and fuzzy open sets of X such that $K \leq U$. Then $f(K)$ is fuzzy closed and $f(U')'$ is fuzzy open set of Y such that $f(K) \leq f(U')'$. Since Y is fuzzy normal, there exists a fuzzy set V of Y such that

$$f(K) \leq \text{int}\ V \leq cl\ V \leq f(U')'.$$

By Corollary 5.9, f is fuzzy $I(so, o)$and injective, we have

$$K = f^{-1}(f(K)) \leq f^{-1}(\text{int}\ V) \leq S - \text{int}\, f^{-1}(V) \leq S - cl\, f^{-1}(V)$$
$$\leq f^{-1}(clV) \leq f^{-1}(f(U')') = U.$$

Thus X is fuzzy S-normal.

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CERTAIN RESULAS FOR A GENERAL CLASS OF POLYNOMIALS, KONHAUSER BIORTHOGONAL POLYNOMIALS AND THE MULTIVARIABLE *H*-FUNCTION

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ABSTRACT

In this paper first we evaluate an integral involving the product of a general classes of polynomials, Konhauser biorthogonal polynomials and the multivariable *H*-function. This integral is then employed to establish an expansion formula for the product of general class of polynomials and the multivariable *H*-function in a series of biorthogonal polynomials. The results established here are quite general in character and a number of (known and new results which follow as special cases of our results are discussed briefly.

1. Introduction

Konhauser ([5], p. 303, 304) has considered following pair of biorthogonal polynomials

$$Z_n^{\alpha}(x;k)=\frac{\Gamma(\alpha+kn+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(\alpha+kj+1)} \qquad ...(1.1)$$

and

$$Y_n^{\alpha}(x;k)=\frac{1}{n!}\sum_{w=0}^{n}\frac{x^w}{w!}\sum_{j=0}^{w}(-1)^j\binom{w}{j}\left(\frac{\alpha+j+1}{k}\right)_n \qquad ...(1.2)$$

which were actually suggested by the Laguerre polynomials, k being a positive integer. Indeed for $k=1$, each of these polynomials reduces to the Laguerre polynomials $L_n^{(\alpha)}(x)$, and their special cases when $k=2$ were encountered earlier by Spencer and Fano [9] in certain analytical calculations involving the penetration of Gamma rays through matter and were studied subsequently by Preiser [6]. (See also Srivastava [10] for comments on the claimed generalizations of the Konhauser polynomials by Raizanda [7, p. 110].)

Srivastava ([11], p.1, Eq. (1)) introduced a general class of polynomials

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$$S_N^M[x] = \sum_{j=0}^{[N/M]} \frac{(-N)_{Mj}}{j!} A_{N,j}\, x^j, \quad N = 0, 1, 2, \ldots \qquad \ldots (1.3)$$

where M is arbitrary positive integer and $A_{N,j}$ $(N, j \geq 0)$ are arbitrary constants, real or complex.

By suitably specializing the coefficients $A_{N,j}$, the general class of polynomials can be reduced to classical orthogonal polynomials, Bessel polynomials and generalized hypergeometric polynomials (see, for example, Srivastava [11] and Srivastava and Singh [12]).

The multivariable H-function occuring in this paper has been defined by Srivastava and Panda ([13], p. 271, Eq. (4.1)). We shall use the following contracted notation (Srivastava et al. [14], p. 251, Eq. (C.1))

$$H[z_1, \ldots, z_u] = H^{O, N : m_1, n_1; \ldots; m_u, n_u}_{P, Q, : p_1, q_1; \ldots; p_u, q_u} \left[\begin{matrix} z_1 \\ \vdots \\ z_u \end{matrix} \right| \left. \begin{matrix} (a_j; \alpha_j', \ldots, \alpha_j^{(u)})_{1,p} : (c_j', \gamma_j')_{1,p_1}; \ldots; (c_j^{(u)}, \gamma_j^{(u)})_{1,p_u} \\ (b_j; \beta_j', \ldots, \beta_j^{(u)})_{1,Q} : (d_j', \delta_j')_{1,q_1}; \ldots; (d_j^{(u)}, \delta_j^{(u)})_{1,q_u} \end{matrix} \right] \qquad \ldots (1.4)$$

to denote the H-function of u complex variables $z_1, \ldots, z_u$. Here all the Greek letters are assumed to be positive real numbers for standardization purposes, the definition of the multivariable H-function will, however, be meaningful even if some of these quantities are zero. The details of these quantities are zero. The details of this function can be found in the papers and book referred to above.

Raizada ([7], p.64, Eq. (2.1.2)) has introduced the generalised polynomial set defined by the following Rodirigues type formula

$$S_n^{\alpha, \beta, \tau}[x; r, s, q, A, B, k, l] = (Ax + B)^{-\alpha} (1 - \tau x^r)^{-\beta/\tau}$$

$$T_{k,l}^{m+n} [(Ax + B)^{\alpha + qn} (1 - \tau x^r)^{(-\beta/\tau) + sn}] \qquad \ldots (1.5)$$

where the differential operator $T_{k,l}$ is defined as

$$T_{k,l} \equiv x^l (k + xD_x) \qquad \ldots (1.6)$$

The explicit form of this generalized polynomial set is ([7], p. 71, Eq. (2.3.4))

$$S_n^{\alpha, \beta, \tau}[x; r, s, q, A, B, m, k, l] = B^{qn} x^{l(m+n)} (1 - \tau x^r)^{sn}\, l^{m+n}$$

$$\sum_{p=0}^{m+n} \sum_{t=0}^{p} \sum_{\delta=0}^{m+n} \sum_{i=0}^{\delta} \frac{(-1)^\delta (-p)_t (-\delta)_i}{p!\, \delta!\, i!} \frac{(\alpha)_\delta}{(\alpha - \delta)_i} \frac{(\alpha - qn)_i}{i!}$$

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$$(-\frac{\beta}{\tau} - sn)_p \, (\frac{i+k+rt}{l})_{m+n} \, (\frac{-\tau x^r}{1-\tau x^r})^p \, (\frac{Ax}{B})^\delta \qquad \ldots (1.7)$$

Taking $A = 1, B = 0$ and $\tau = 0$ in (1.7), we arrive at the following polynomial set :

$$\lim_{\tau \to 0} S_n^{\alpha, \beta, \tau} [x; r, s\, q, 1, 0, m\, k, l]$$

$$= S_n^{\alpha, \beta, 0} [x; r, q, 1, 0, m, k, l]$$

$$= x^{qn + l(m+n)} \, l^{m+n} \sum_{p=0}^{m+n} \sum_{t=0}^{p} \frac{(-p)_t}{p!\, t!} \left(\frac{\alpha + qn + k + rt}{l} \right)_{m+n} (\beta x^r)^p \qquad \ldots (1.8)$$

The above-mentioned polynomial set (1.8) is also general in nature and contains known polynomials due to Gould and Hopper [2], Singh and Srivastava [8], Chatterjee [1] and Krall and Frink [3] as its special cases.

2. Main Integral : In this section we evaluate the following general integral

$$\int_\xi^\infty (x-\xi)^{\mu-1} e^{-\eta x} \, Y_n^\alpha (x; k) \, S_\lambda^{\nu, \zeta, 0} (y(x-\zeta)^\rho ; r, q, 1, 0, m, k, l)$$

$$S_N^M [z(x-\xi)^\sigma] \, H[z_1 (x-\xi)^{\sigma_1}, \ldots, z_h (x-\xi)^{\sigma_h}] \, dx$$

$$= e^{-\eta\xi} \, e^{m+\lambda} \, y^{q\lambda + l(m+\lambda) + rp} \sum_{p=0}^{m+\lambda} \sum_{f=0}^{p} \sum_{\delta=0}^{[N/M]} \sum_{w=0}^{n} \sum_{i=0}^{w}$$

$$\frac{(-p)_f (-N)_{M\delta} \, (-w)_i}{p!\, f!\, \delta!\, i!\, n!\, w!} \, A_{N,\delta} \left(\frac{\nu + q\lambda + k + rf}{l} \right)_{m+\lambda} (-1)^n \, z^\delta \, \zeta^p \, \xi^w$$

$$H_{i, \delta, p, t}^{\mu'} [z_1, \ldots, z_h] \qquad \ldots (2.1)$$

where $\mu' = \mu + q\lambda\rho + \rho l(m+\lambda) + \sigma\delta + \rho rp$ $\qquad \ldots (2.2)$

and

$$H_{i, \delta, p, t}^{\mu'} [z_1, \ldots, z_h]$$

$$= H^{0, B+2 : A_1, B_1 ; \ldots ; A_h, B_h}_{C+2, D+1 : C_1, D_1 ; \ldots ; C_h, D_h} \left[\begin{matrix} z_1 \\ \vdots \\ z_h \end{matrix} \right| \begin{matrix} (1-\mu'; \sigma, \ldots, \sigma_h), \\ (\frac{1+i+\alpha-\mu'}{k} + n ; \frac{\sigma_1}{k}, \ldots, \frac{\sigma_h}{k}), \end{matrix}$$

$$\begin{matrix} \frac{1+i+\alpha-\mu'}{k} ; \frac{\sigma_1}{k}, \ldots, \frac{\sigma_h}{k}), (a_j, \alpha_j', \ldots, \alpha_j^{(h)})_{1, C} ; \\ (b_j ; \beta_j', \ldots, \beta_j^{(h)})_{1, D} : \end{matrix}$$

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$$\left. \begin{matrix} (c_j', \gamma_j')_{1, c_1}; \ldots; (c_j^{(h)}, \gamma_j^{(h)})_{1, C_h} \\ (d_j', \delta_j')_{1, D_1}; \ldots; (d_j^{(h)}, \delta_j^{(h)})_{1, D_h} \end{matrix} \right] \quad \ldots (2.3)$$

The (sufficient) conditions of validity of (2.1) are

(i) k, s and r are positive integers.

(ii) $\mathrm{Re}(\eta) > 0$, $\mathrm{Re}(\alpha + 1) > 0$,

$$\mathrm{Re}(\mu + \sigma\delta) + \sum_{j=1}^{h} \sigma_j \min_{1 \le i \le A_j} \mathrm{Re}[d_i^{(j)}/\delta_i^{(j)}] > 0, \ (\delta = 1, \ldots, [N/M])$$

(iii) $\Omega_j > 0$, $|arg\, z_i| < 1/2\, \Omega_i\, \pi$

where

$$\Omega_j = -\sum_{i=B+1}^{C} \alpha_i^{(j)} - \sum_{i=1}^{D} \beta_i^{(j)} + \sum_{i=1}^{A_j} \delta_i^{(j)} - \sum_{i=A_j+1}^{D_j} \delta_i^{(j)}$$

$$+ \sum_{i=1}^{B_j} \gamma_i^{(j)} - \sum_{i=B_j+1}^{C_j} \gamma_i^{(j)} \quad (j = 1, 2, \ldots, h) \quad \ldots (2.4)$$

Proof

To evaluate the integral (2.1), we first express the multi-variable H-function involved in the left-hand side in terms of multiple Mellin-Barnes type contour integral with the help of (1.4), general class of polynomials and generalised polynomial set in series given by (1.3) and (1.8) respectively and change the order of integrations, wich is permissible under the conditions stated with (2.3), we get the left-hand side of (2.1) as

$$\Delta = \sum_{p=0}^{m+\lambda} \sum_{f=0}^{p} \sum_{\delta=0}^{[N/M]} \frac{(-p)_f}{p!\, f!} \zeta^p \frac{(-N)_{M\delta}}{\delta!} A_{N,\delta} \left(\frac{\nu + q\lambda + k + rf}{l} \right)_{m+\lambda}$$

$$l^{m+\lambda} z^{\delta} y^{q\lambda + l(m+\lambda) + rp}$$

$$\frac{1}{(2\pi i)^h} \int_{L_1} \ldots \int_{L_h} \theta(t_1) \ldots \theta(t_h)\, \psi(t_1, \ldots, t_h)$$

$$\left\{ \int_{\xi}^{\infty} (x - \xi)^{\mu + q\lambda\rho + l\rho(m+\lambda) + \sigma\delta + \rho rp + \sum_{j=1}^{h} \sigma_j t_j - 1} \right.$$

$$\left. e^{-\eta x}\, Y_n^{\alpha}(x; k)\, dx \right\} z_1^{t_1} \ldots z_h^{t_h}\, dt_1 \ldots dt_h \quad \ldots (2.5)$$

Now for evaluation of the inner x-integral, using the {[10], p. 43, Eq. (3.9)), changing the order of integration and summation involved therein and expressing the multiple contour integral as the multi-variable H-function, we easily get the right-had side of (2.1).

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3. Special cases of (2.1)

(i) Taking $z = M = 1, \sigma = g, A_{N,\delta} = \dfrac{\Gamma(1+\beta+gN)}{N!\,\Gamma(1+\beta+g\delta)}$

and using the following relationship

$S_N^1[(x-\xi)^g] \to Z_N^\beta[x-\xi); g)$ in (2.1), we get the follwoing interesting integral :

$$\int_\xi^\infty (x-\xi)^{\mu-1} e^{-\eta x} Y_n^\alpha(x; k)\, Z_N^\beta((x-\xi); g)$$

$$S_\lambda^{\nu,\zeta,0}(y(x-\xi)^\rho; r, q, 1, 0, m, k, l)\, H[z_1(x-\xi)^{\sigma_1}, \ldots, z_h(x-\xi)^{\sigma_h}]\, dx$$

$$= e^{-\eta\xi} l^{m+\lambda} y^{q\lambda+(m+n)+rp} \sum_{p=0}^{m+\lambda} \sum_{f=0}^{p} \sum_{\delta=0}^{N} \sum_{w=0}^{n} \sum_{i=0}^{w}$$

$$\frac{(-p)_f (-N)_\delta (-w)_i\, \Gamma(1+\beta+gN)}{p!\,f!\,\delta!\,n!\,w!\,N!\,\Gamma(1+\beta+g\delta)} \left(\frac{\nu+q\lambda+k+rf}{l}\right)_{m+\lambda}$$

$$(-1)^n \zeta^p \xi^w H_{i,\delta,p,t}^{\mu''}[z_1, \ldots, z_h] \qquad \ldots (3.1)$$

where $\mu'' = \mu + q\lambda l + \rho l(m+\lambda) + g\delta + \rho rp.$ ... (3.2)

(ii) Taking $M = 1, N = 0, A_{N,\delta} = 1$ in (2.1), then $S_0^1[z(x-\xi)^\sigma]$ reduce to one and the integral (2.1) takes the following form

$$\int_\xi^\infty (x-\xi)^{\mu-1} e^{-\eta x} Y_n^\alpha(x; k)\, S_\lambda^{\nu,\zeta,0}(y(x-\xi)^\rho; r, q, 1, 0, m, k, l)$$

$$H[z_1(x-\xi)^{\sigma_1}, \ldots, z_h(x-\xi)^{\sigma_h}]\, dx$$

$$= e^{-\eta\xi} l^{m+\lambda} y^{q\lambda+l(m+\lambda)+rp} \sum_{p=0}^{m+\lambda} \sum_{f=0}^{p} \sum_{w=0}^{n} \sum_{i=0}^{w}$$

$$\frac{(-p)_f (-w)_i}{p!\,f!\,n!\,i!\,w!} \left(\frac{\nu+q\lambda+k+rf}{l}\right)_{m+\lambda} (-1)^n \zeta^p \xi^w H_{i,o,p,t}^{\mu'''}[z_1, \ldots, z_h]$$

... (3.3)

where $\mu''' = \mu + q\lambda\rho + \rho l(m+\lambda) + \rho rp.$

4. Expansion theorem

Suppose

(i) k and r are positive integers,

(ii) $\sigma_j > 0, \sigma > 0, \mu > 0;\ \mathrm{Re}(\alpha) > -1,$

$$\mathrm{Re}(\beta+\sigma\delta) + \sum_{j=1}^{h} \sigma_j \min_{1<i\le A_j} \mathrm{Re}(d_i^{(j)}/\delta_i^{(j)}) > -1$$

$$(\delta = 1, \ldots, [N/M])$$

(iii) $\Omega_j > 0 \quad |arg\, z_j| < 1/2\, \Omega_j \pi \ (j = 1, \ldots, h)$

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(Ω_j is defined by (2.4)),

then

$$x^{\mu-1} S_{\lambda}^{\nu, \zeta, 0}(y x^{p}; r, q, 1, 0, m, k, l)\, S_N^M [z x^{\sigma}]\, H[z_1 x^{\rho_1}, ..., z_h x^{\sigma_h}]$$

$$= \sum_{n=0}^{\infty} \left[l^{m+\lambda} y^{q\lambda + l(m+\lambda) + rp} \sum_{p=0}^{m+\lambda} \sum_{f=0}^{p} \sum_{\delta=0}^{[N/M]} \sum_{w=0}^{n} \sum_{i=0}^{w} \right.$$

$$\frac{(-p)_f (-N)_{M\delta} (-w)_i}{p!\, f!\, \delta!\, w!\, i!} \left(\frac{\nu + q\lambda + k + rf}{l}\right)_{m+\lambda} A_{N,\delta}\, \zeta^{p} z^{\delta} \frac{(-1)^n}{\Gamma(1 + \alpha + kn)}$$

$$\left. H_{i, \delta, p, t}^{\mu' + \alpha} [z_1, ..., z_h] \right] Z_n^{\alpha}(x; k) \qquad \ldots (4.1)$$

Proof : Let

$$x^{\mu-1} S_{\lambda}^{\nu, \zeta, 0}(y x^{\rho}; r, q, 1, 0, m, k, l)\, S_N^M [z x^{\sigma}]\, H[z_1 x^{\sigma_1}, ..., z_h x^{\sigma_h}]$$

$$= \sum_{n=0}^{\infty} k_n Z_n^{\alpha}(x; k) \qquad \ldots (4.2)$$

Multiplying both sides of (4.2) by $e^{-x} x^{\alpha} Y_{\upsilon}^{\alpha}(x; k)$ and intergrating with respect to x from 0 to ∞, we get

$$\int_0^{\infty} e^{-x} x^{\mu+\alpha-1} Y_k^{\alpha}(x; k)\, S_{\lambda}^{\nu, \zeta, 0}(y x^{\rho}; r, q, l, 0, m, k, l)$$

$$S_N^M [z x^{\sigma}]\, H[z_1 x^{\sigma_1}, ..., z_h x^{\sigma_h}]\, dx$$

$$= \sum_{n=0}^{\infty} k_n \int_0^{\infty} e^{-x} x^{\alpha} Z_n^{\alpha}(x; k)\, Y_{\upsilon}^{\alpha}(x; k)\, dx \qquad \ldots (4.3)$$

using the integral (2.1) and the following orthogonal property ([5], p. 303)

$$\int_0^{\infty} x^{\alpha} e^{-x} Y_{\upsilon}^{\alpha}(x; k)\, Z_n^{\alpha}(x; k)\, dx = \frac{\Gamma(\alpha + kn + 1)}{n!}\, \delta_{\upsilon, n} \qquad \ldots (4.4)$$

(where $\mathrm{Re}(\alpha) > -1$, k, υ, n are positive integers. Also $\delta_{\upsilon, n}$ is the well-known Kronecker delta function)

in (4.3), we find that

$$K_n = l^{m+\lambda} y^{q\lambda + l(m+\lambda) + rp} \sum_{p=0}^{m+\lambda} \sum_{f=0}^{p} \sum_{\delta=0}^{[N/M]} \sum_{w=0}^{n} \sum_{i=0}^{w}$$

$$\frac{(-p)_f (-N)_{M\delta} (-w)_i}{p!\, f!\, \delta!\, i!\, w!} \left(\frac{\nu + q\lambda + k + rf}{l}\right)_{m+\lambda} A_{N,\delta}\, \zeta^{p} \frac{(-1)^n z^{\delta}}{\Gamma(1 + \alpha + kn)}$$

$$H_{i, \delta, p, t}^{\mu' + \alpha} [z_1, ..., z_h] \qquad \ldots (4.5)$$

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where $H^{\mu'+\alpha}_{i,\delta,p,t}[z_1, ..., z_h]$ can be defined similarly to (2.3).

Substituting the value of K_n from (4.5) in (4.2), we arrive at the required expansion formula (4.1).

5. Special Cases of (4.1) : Using the substitutions in (4.1) as mentioned with the special cases of (2.1), we arrive easily at the following expansion formulas

$$x^{\mu-1} Z^{\beta}_{N}(x; g)\, S^{\nu,\zeta,0}_{\lambda}(y x^{\rho}; r, q, 1, 0, m, k, l)\, H[z_1 x^{\sigma_1}, ..., z_h x^{\sigma_h}]$$

$$= l^{m+\lambda} y^{q\lambda+l(m+\lambda)+rp} \sum_{n=0}^{\infty} \sum_{p=0}^{m+\lambda} \sum_{f=0}^{p} \sum_{\delta=0}^{N} \sum_{w=0}^{n} \sum_{i=0}^{w}$$

$$\frac{(-p)_f (-N)_\delta (-w)_i}{p!\, f!\, \delta!\, w!\, i!} \left(\frac{\nu+q\lambda+k+rf}{l}\right)_{m+\lambda} \frac{\Gamma(1+\beta+gN)}{N!\,\Gamma(1+\beta+g\delta)} \frac{(-1)^n}{\Gamma(1+\alpha+kn)} \zeta^p$$

$$H^{\mu'+\alpha}_{i,\delta,p,t}[z_1, ..., z_n] \qquad \text{... (5.1)}$$

and

$$x^{\mu-1} S^{\nu,\zeta,0}_{\lambda}(yx^{\rho}; r, q, 1, 0, k, m, l)\, H[z_1 x^{\sigma_1}, ..., z_h x^{\sigma_h}] =$$

$$= l^{m+\lambda} y^{q\lambda+l(m+\lambda)} \sum_{n=0}^{\infty} \sum_{p=0}^{m+\lambda} \sum_{f=0}^{p} \sum_{w=0}^{n} \sum_{i=0}^{w}$$

$$\frac{(-p)_f (-w)_i}{p!\, f!\, w!\, i!} \left(\frac{\nu+q\lambda+k+rf}{l}\right)_{m+\lambda} \frac{(-1)^n}{\Gamma(1+\alpha+kn)} \zeta^p$$

$$H^{\mu'+\alpha}_{i,0,p,t}[z_1, ..., z_h] \qquad \text{... (5.2)}$$

The conditions of validity for (5.1) and (5.2) are easly obtainable from those mentioned with the main expansion theorem (4.1).

A number of other integrals and expansion formulas involving product of elementary special functions of one and more variables can be obtained from (2.1), (3.1), (3.2), (4.1), (5.1) and (5.2) as special cases. This can be done by specializing the parameters of the multi-variable H-function in a suitable manner.

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(Dedicated to Professor J.N. Kapur on his 70th Birthday)

A GENERALIZED STUDY OF A VISCOUS INCOMPRESSIBLE FLUID FLOW THROUGH VARIOUS CROSS SECTIONS OF A TUBE

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ABSTRACT

An entirely new technique has been developed to study the velocity disrribution of viscous fluids flowing through tubes whose cross sections are any rectilinear figure of n-sides.

1. INTRODUCTION

The number of cases of viscous incompressible fluids flowing through channels of various cross section viz, rectangle, equilateral triabgle, right angle triangle, between parallel plates, elliptic section of tubes [1] have been discussed by various investigators. Besides these, the velocity distribution over the cross section of tube has also been determined for section bounded by (i) confocal ellipses (ii) a line and a parabola (iii) two confocal parabolas and part of their common axis (iv) two confocal parabolas etc. by different investigators [1]. In this paper an entirely new technique has been developed to study the velocity distribution when the cross section is any rectilinear figure of n sides.

2. PRELIMINARIES

In case of steady motion along a pipe of uniform cross section, if z-axis is taken parallel to the length of the pipe;

$$u = v = 0;$$

$$\frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 0.$$

So Navier Stokes equations reduce to a single equation

$$\rho z - \frac{\partial p}{\partial z} + \mu \nabla^2 w = 0. \tag{2.1}$$

Assuming z to be zero and the pressure gradient $\frac{\partial p}{\partial z}$ to be constant $(= -h)$, the equation (2.1) reduces to,

$$\nabla^2 w = -\frac{h}{\mu} = -K \text{ (say)}. \tag{2.2}$$

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Hence, finally we get,

$$\textbf{(2.3)} \qquad \frac{\nabla^2 w}{\partial x^2} + \frac{\nabla^2 w}{\partial y^2} = -K$$

which has a particular solution

$$\textbf{(2.4)} \qquad w = -\frac{K}{2}\frac{Ax^2 + By^2}{A+B}.$$

If $\Phi(x, y)$ is also a solution of $\nabla^2 \varphi(x, y) = 0$. Then (2.3) has a general solution

$$\textbf{(2.5)} \qquad w = \varphi(x, y) - \frac{K}{2}\frac{Ax^2 + By^2}{A+B}.$$

When the liquid touches the wall of the pipe, $w = 0$ and we have

$$\textbf{(2.6)} \qquad \varphi(x, y) - \frac{K}{2}\frac{Ax^2 + By^2}{A+B} = 0$$

which gives the boundry of the cross section of the pipe for flow through which **(2.5)** holds.

$\varphi(x, y)$ is circular harmonics which are known to be of one of the froms of the harmonics $r^n \cos(n\theta)$, $r^n \sin(n\theta)$ $\frac{\cos(n\theta)}{r^n}$, $\frac{\sin(n\theta)}{r^n}$, $\log r$ and θ or their combinations.

In cartesian coordinates the harmonics are given by

$x^n - {}^nc_2\, x^{n-2}y^2 + {}^nc_4 x^{n-4}y^4 \ldots$ and ${}^nc_1 x^{n-1}y - {}^nc_3 x^{n-3}y^3 \ldots$,

which correspond to $r^n\cos(n\theta)$ and $r^n\sin(n\theta)$. Dividing them by $(x^2+y^2)^n$, we get the forms corresponding to $\frac{\cos(n\theta)}{r^n}$ and $\frac{\sin(n\theta)}{r^n}$.

The harmonics of the type $\log r$ and θ in polar coordinates are given by $\log(x^2+y^2)$ and $\tan^{-1}(\frac{y}{x})$ in cartesian coordinates.

These harmonics result from the complex transformation,

$$w = z^n \text{ and } w = \log z,$$

where n is positive or negative integer. Unlimited number of harmonics are derived from the transformation $w = f(z)$.

First case of interest occurs when $\varphi(x, y)$ is at most of second degree in (x, y). In this case

$$(2.7) \qquad w = a_2(x^2 - y^2) + 2b_2xy + a_1x + b_1y + a_0 - \frac{K}{2}\frac{Ax^2 + By^2}{A+B}.$$

The boundary is given by

$$\textbf{(2.8)} \qquad \left[a_2 - \frac{KA}{2(A+B)}\right]x^2 + 2b_2xy - \left[a_2 + \frac{KB}{2(A+B)}\right]y^2 + a_1x + b_1y + a_0 = 0$$

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which is a conic section.

If equation **(2.8)** is written as,

(2.9) $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, then

(2.10) $w = ax^2 + 2hxy + 2gx + 2fy + by^2 + c$

gives the velocity function admissible. The point of maximum velocity coincides with the centre because they are given by the same set of equations.

If (x', y') is the centre, then the velocity at centre is,

(2.11) $w_c = gx' + fy' + c.$

Constants a, b and K are related by the equations;

$$\nabla^2 w = 2a + 2b = -K,\ a + b = -\frac{K}{2}.$$

For special values of the constants the general form (2.10) assumes one of the following forms :

(2.12) $$w = -\frac{K}{2}(x^2 + y^2 - \alpha^2)$$

when the cross section is the circle, $x^2 + y^2 = \alpha^2$

(2.13) $$w = -\frac{K}{2}\frac{a^2b^2}{a^2 + b^2}\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right]$$

when the cross section is ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(2.14) $$w = \frac{K}{2}\frac{a^2b^2}{b^2 - a^2}\left[\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right]$$

when the cross sectiion is hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is not a closed curve therefore channels must be constructed instead of pipes.

Flow in parabolic channels may be studied by the form

(2.15) $$w = -\frac{K}{2}(x^2 - 4y)$$

while the flow between two parallel planes may be formulated in the forms

(2.16) $$w = \frac{K}{2}(x^2 - a^2)$$

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(2.17) $w = \frac{K}{2}(x^2 - mxy)$ or more symmetrical form

(2.18) $w = \frac{k}{2(1-m^2)}(x^2 - m^2y^2)$

gives the flow in V-shaped channels of two intersecting planes.

These special forms and many others have been studied in detail by many authors. Harmonics of the type cosh (nx) cos (ny) in series with suitable coefficients have been employed to form velocity fuction for flow when the cross section is rectangle $x = \pm a, y = \pm b$, and also when it is right angled triangle. Third degree harmonics have been employed for equilateral triangle. But the case of any triangle or any parallelogram or other rectilinear figure have not been solved as yet.

In what follows, we propose to develope an entirely new technique and find out the velocity distribution when the cross section is any rectilinear figure of n-sides. We shall limit our selves to quadratic form of the velocity function and shall not use higher degree harmonics nor harmonics like cosh (nx) cosh (ny).

3. BOUNDARY OF THE RECTILINEAR FIGURE OF n-SIDES

The polar equation of straight line is

(3.1) $\rho \cos(\theta - \alpha) = p.$

If there are n lines, their equation may be written as,

(3.2) $\rho \cos(\theta - \alpha_r) = p, \qquad (r = 1, 2, 3, \ldots, n)$

If β_r is the angular coordinate of the point of intersection of rth and $(r+1)^{th}$ lines then β_r satisfies,

(3.3) $\cos(\beta_r - \alpha_r)p_{r+1} = \cos(\beta_r - \alpha_{r+1})p_r$.

The rth line extends from

$\theta = \beta_{r-1}$ to $\theta = \beta_r$.

Since the figure is closed, $\beta_0 = \beta_n$

The figure under consideration may be said to have a boundary given by,

(3.4) $\rho \cos(\theta - \alpha_r) = p_r$; $\beta_{r-1} < \theta < \beta_r$.

This boundary may be represented more elegantly by means of functions which we shall call $\phi(\theta, \beta\beta')$; $\beta < \beta'$

$\phi(\theta, \beta\beta')$, have the following definition :

$\Phi(\theta, \beta\beta') = 1$ for $\beta < \theta < \beta'$

$\phi(\theta, \beta\beta') = 0$ for $\beta' < \theta < 2\pi + \beta$.

Hence the whole boundary is given by,

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$$\textbf{(3.5)} \qquad \sum_{r=1}^{r=n} [\rho \cos(\theta - \alpha_r) - p_r] \phi(\theta, \beta_{r-1}\beta_r) = 0.$$

The same boundary is also given by,

$$\textbf{(3.6)} \qquad \sum_{r=1}^{r=n} \lambda_r [\rho \cos(\theta - \alpha_r) - p_r] \phi(\theta, \beta_{r-1}\beta_r) = 0$$

where λ_r are arbitrary constants.

In what follows a quadratic form is preferable for the boundary, so in place of **(3.5)** or **(3.6)** we shall denote the boundary by,

$$\textbf{(3.7)} \qquad \sum_{r=1}^{r=n} [p_r^2 - \rho^2 \cos^2(\theta - \alpha_r] \phi(\theta, \beta_{r-1}\beta_r) = 0$$

$$\textbf{(3.8)} \qquad \sum_{r=1}^{r=n} \lambda_r [p_r^2 - \rho^2 \cos^2(\theta - \alpha_r)] \phi(\theta, \beta_{r-1}\beta_r) = 0$$

for $\beta_{r-1} < \theta < \beta_r$ in each case.

In fact $p^2 = \rho^2 \cos^2(\theta - \alpha)$, represents pair of lines both being parallel and on opposit sides of the origin at distance p. Therefore (3.7) or (3.8) represent two equal rectilinear figures each of which is obtainable by rotating the other round the origin through 180^0. This fact will have to be taken into account in calculating total flow per unit time.

It should be noted that functions like $\phi(\theta, \beta\beta')$ are always possible to construct and these may be analytically expressed for $0 < \theta < 2\pi$ as,

$$\Phi(\theta, \beta\beta') = \sum_{s=0}^{s=\infty} C_s \cos(s\theta)$$

where C_s are the Fourier coefficients calculated in the usual manner **[2]**.

The values of $\phi(\theta, \beta\beta')$ relevant for this calculation are :

$\phi = 0$ for $0 < \theta < \beta$

$\phi = 1$ for $\beta < \theta < \beta'$

$\phi = 0$ for $\beta' < \theta < 2\pi$

Therefore

$$C_s = \frac{2}{\pi} \int_0^{\pi} \phi(\theta, \beta\beta') \cos(s\theta)\, d\theta$$

$$= \frac{2}{\pi} \int_{\beta}^{\beta'} \cos(s\theta)\, d\theta$$

$$= \frac{2}{s\pi} (\sin s\beta' - \sin s\beta) \text{ and}$$

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$$C_0 = \frac{1}{\pi}\int_{\beta}^{\beta'} d\theta = \frac{\beta' - \beta}{\pi}.$$

Thus equation (3.8) written in full is,

$$\textbf{(3.9)}\quad \sum_{r=1}^{r=n} \lambda_r \left[p_r^2 - \rho^2 \cos^2(\theta - \alpha_r)\right]\left[\frac{\beta_r - \beta_{r-1}}{\pi} + \sum_{s=1}^{s=\infty} \frac{2(\sin s\beta_r - \sin s\beta_{r-1})}{s\pi}\right].$$

4. DETERMINATION OF THE VELOCITY DISTRIBUTION

Here we shall be studying the velocity distribution given by,

$$\textbf{(4.1)}\quad w = \sum_{r=1}^{r=n} \frac{K}{2}\left[p_r^2 - \rho^2 \cos^2(\theta - \alpha_r)\right] \phi(\theta, \beta_{r-1}\, \beta_r)$$

For $\beta_{r-1} < \theta < \beta_r$

$$w = \frac{K}{2}\left[p_r^2 - \rho^2 \cos^2(\theta - \alpha_r)\right]$$

$$w = \frac{K}{2}\left[p_r^2 - x^2 \cos^2\alpha_r - y^2 \sin^2\alpha_r - 2xy \cos\alpha_r \sin\alpha_r\right]$$

so that,

$$w = \frac{K}{2}\left[-2\cos^2\alpha_r - 2\sin^2\alpha_r\right] = -K.$$

Also $w = 0$ gives the boundary considered in 3 i.e.

$$\sum_{r=1}^{r=n}\left[p_r^2 - \rho^2 \cos^2(\theta - \alpha_r)\right] \phi(\theta, \beta_{r-1}\, \beta_r) = 0.$$

Thus equations (4.1) give the form of velocity function for steady flow of an incompressible viscous fluid in pipe whose cross section is rectilinear figure of n sides.

So far we have taken p_r to be quite arbitrary but continuity considerations impose limitations on the value of p_r which we now proceed to dicuss.

The velocity function should be continuous from both the sides of the line $\theta = \beta_r$, so,

$$\textbf{(4.2)}\quad \frac{K}{2}\left[p_r^2 - \rho^2 \cos^2(\beta_r - \alpha_r)\right] = \frac{K}{2}\left[p_{r+1}^2 - \rho^2 \cos^2(\beta_r - \alpha_{r+1})\right]$$

for all values of ρ.

This is possible only if

$p_r = p_{r+1}$ and $\beta_r - \alpha_r = \alpha_{r+1} - \beta_r$. So we get,

$$\textbf{(4.3)}\quad p_1 = p_2 = p_3 = \ldots = p_n \text{ and every}$$

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(4.3) $p_1 = p_2 = p_3 = \ldots = p_n$ and every

(4.4) $$\beta_r = \frac{\alpha_r + \alpha_{r+1}}{2}$$

In fact two conditions (4.3) and (4.4) are identical, when one is satisfied, the other will also be satisfied. Under these conditions w takes the form,

(4.5) $$w = \sum_{r=1}^{r=n} \frac{K}{2}\left[p^2 - \rho^2 \cos^2(\theta - \alpha_r)\right]\left(\theta, \frac{\alpha_{r-1} + \alpha_r}{2} \frac{\alpha_r + \alpha_{r+1}}{2}\right)$$

or more simply for

$$\frac{\alpha_{r-1} + \alpha_r}{2} < \theta < \frac{\alpha_r + \alpha_{r+1}}{2}$$

(4.6) $$w = \frac{K}{2}\left[p^2 - \rho^2 \cos^2(\theta - \alpha_r)\right].$$

The boundary of the cross section of the pipe being formed of lines,

(4.7) $\rho \cos(\theta - \alpha_r) = \pm p;\ r = 1, 2, 3, \ldots n.$

The rectilinear figure given by (4.7) has the following properties :

(1) A circle can be inscribed in the figure touching all the sides.

(2) Origin or $\rho = 0$, is the centre of the circle.

(3) p is the radius of the in-circle. So that all the boundary lines are equidistant from the origin and all such figures can be described round a circle.

If L is the perimeter of the rectilinear figure then the perimeter of a similar figure indside with sides at distance of ξ from the outer side is,

$$\frac{L(p - \xi)}{p}.$$

So that the whole area of cross section is given by,

(4.8) $$A = \int_0^p \frac{L(p - \xi)}{p} d\xi = \frac{Lp}{2}$$

and total flow per unit time,

$$Q = \int_0^p \frac{K}{2}\left[p^2 - (p - \xi)^2\right] \frac{L(p - \xi)}{p} d\xi = \frac{KLp^3}{8}.$$

This must be the flow in pair represented by (3.7) or (3.8) therefore for a single pipe we get,

(4.9) $$Q = \frac{Klp^3}{16}.$$

Mean velocity

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$$U = \frac{Q}{A} = \frac{KLp^3}{16}\frac{2}{Lp} = \frac{Kp^2}{8}. \tag{4.10}$$

Maximum velocity

$$c = \frac{Kp^2}{2} \tag{4.11}$$

$$k = \frac{U}{KA} = \frac{Kp^2}{8}\frac{2}{KLp} = \frac{p}{4L} \tag{4.12}$$

$$k' = \frac{c}{KA} = \frac{2Kp^2}{2KLp} = \frac{p}{L}. \tag{4.13}$$

5. SOME PARTICULAR CASES OF VISCOUS FLOW THROUGH TRIANGULAR AND RECTANGULAR CROSS SECTIONS

Triangles are figures with smallest numbers of sides that can be described round a circle.

The general formula for total flow per unit time for rectilinear figures in section **4** has been obtained as,

$$Q = \frac{KLp^3}{16}$$

If S is the area and s is the perimeter then

$$p = \frac{S}{s} \text{ and } L = 2s$$

Therefore,

$$Q = \frac{2Ks}{16}\frac{S^3}{s} = \frac{KS^3}{8s^2} \tag{5.1}$$

It will be interesting to compare this result (5.1) with two known solutions for triangles given by Boussinesq. Boussinesq formulae for equilateral triangle given by,

$y = \pm\sqrt{3}x, y = b$, gives,

$$Q = \frac{Kb^4}{60\sqrt{3}} \tag{5.2}$$

Here, in this case,

$$S = \frac{b^2}{\sqrt{3}}, s = \sqrt{3}b.$$

Therefore from **(5.1)** we get

$$Q = \frac{K}{8}\frac{b^6}{3\sqrt{3}}\frac{1}{3b^2} = \frac{Kb^4}{72\sqrt{3}}. \tag{5.3}$$

Hence, **(5.2)** and **(5.3)** give values of Q in the ratio of 6:5.

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Solution obtained by Proudman and others for rightangled triangle given by,

$x = a, y = a, x + y = 0$, gives

$$(5.4) \quad Q = \frac{K}{4a}\left[\frac{4}{3}a^5 - \sum_{n=0}^{\infty} N^{-5} \coth(Na)\right]$$

where $2Na = (2n + 1)\pi$.

In this case

$$S = \frac{a^2}{2}; \; s = (1 + 1/\sqrt{2})a.$$

Therefore, Q from **(5.1)** is given by,

$$(5.5) \quad Q = \frac{K\,2a^6}{64(3 + 2\sqrt{2})a^2} = \frac{Ka^4}{186.496}.$$

Thus formula **(5.1)** for Q in case of triangular cross section does not agree with the two known solutions. But when the number of sides is made indefinitely great and the rectilinear figure becomes a circle of radius a ;

$$S = \pi a^2 \text{ and } s = \pi a$$

then from **(5.1)** we get,

$$(5.6) \quad Q = \frac{K\pi^3 a^6}{8\pi^2 a^2} = \frac{\pi k a^4}{8},$$

which agrees with the known solution for pipe of circular cross section. This result **(5.6)** may be taken as a clue to the correctness of the solution obtained in previous section. However, the result can be experimentally checked for the correctness of the formulae of previous section.

In case of a square where sides are $2a$;

$$S = 4a^2, \; s = 4a;$$

Q is given by

$$Q = \frac{64\,K\,a^6}{128\,a^2} = \frac{K\,a^4}{2}$$

and $$k = \frac{1}{32}, \; k' = \frac{1}{8}.$$

In similar way total flow for other cases of rectangle may be calculated. The result of this section can be easily extended to the cases when the cross section of the pipe is made up of arc of conic sections instead of straight lines.

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TWO FIXED POINT THEOREMS FOR NONSELF MAPPINGS

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ABSTRACT

Nadim Assad [1] established two fixed point theorems for nonself maps. In this paper we generalize his contractive definitions and establish the corresponding fixed point theorems.

Nadim Assad [1] established a fixed point theorem for a nonself map satisfying the condition

(1) $d(Tx, Ty) \le b(x, y)\ [d(x, Tx) + d(y, Ty)] + c(x, y) \min\{d(x, Ty), d(y, Tx)\}$,

where b and c are decreasing functions from $R^+ \to [0, 1)$ such that $2b(t) + c(t) < 1$ for all $t > 0$. He also established a fixed point theorem for K compact, T continuous, and strict inequality in (1) with $2b(t) + c(t) = 1$.

The purpose of this paper is to generalize Assad's definition [1], and then prove the corresponding fixed point theorems.

Let $x_0 \in X$, $x_1' := Tx_0$ If $x_1' \in K$, set $x_1' = x_1$. Otherwise, choose $x_1 \in \partial K$ such that $d(x_0, x_1') = d(x_0, x_1) + d(x_1, x'_1)$. Then $x_1 \in K$ and Tx_1 is defined. Suppose that, for $n \ge 1$, $\{x_0, x_1, ...\}$ and $\{x_0', x_1', ...\}$ have been chosen so that, for $1 \le i \le n$,

(*i*) $x_i' = Tx_{i-1}$,

(*ii*) $x'_i = x_i$ if $x_i' \in K$, or

(*iii*) $x_i \in \partial K$ and satisfies the relation

$$d(x_{i-1}, x_i') \ ... = d(x_{i-1}, x_i) + d(x_i, x_i').$$

Let $x'_{n+1} = Tx_n$. If $x'_{n+1} \in K$, set $x_{n+1} = x'_{n+1}$. Otherwise choose $x_{n+1} \in \partial K$ so that $d(x_n, x'_{n+1}) = d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1})$. The sequecne $\{x_n\}$ so constructed shall be called the general orbit of T at x_0.

Let $P := \{x_i \in \{x_n\} : x_i = x_i'\}$ and $Q := \{x_i \in \{x_n\} : x_i \ne x_i'\}$. Note that, if $x_n \in Q$, then $x_{n+1} \in P$.

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Theorem 1. *Let X be a Banach space, K a nomempty closed subset of X, $T : K \to X$ satisfying the condition that $x \in \partial K$ implies that $Tx \in K$ and, for all $x, y \in X$,*

$$(2) \qquad d(Tx, Ty) \le b(x, y) \max \{d(x, Tx), d(y, Ty)\} + c(x, y) \min \{d(x, Ty), d(y, Tx)\}$$

where b and c are decreasing functions from $R^+ \to [0, 1)$ such that $b(t) + c(t) < 1$ for all $t > 0$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$, $\{x_n\}$ a general orbit of x_0, $\tau_n := d(x_n, x_{n+1})$. Without any loss of generality we may assume that $\tau_n > 0$ for each n. For, if there exists a value of n such that $\tau_n = 0$, then $x_n = x_{n+1}$. If $x_n \in \partial K$, then $x'_{n+1} \in K$ so that $x'_{n+1} = x_{n+1}$. Thus $x_n = x_{n+1} = Tx_n$ and x_n is a fixed point of T. If $x_n \notin \partial K$, then $x_n \in K$ and either $x'_{n+1} \in K$ or $x'_{n+1} \notin K$. If $x'_{n+1} \in K$, then, as before, $x_n = x_{n+1} = Tx_n$, and x_n is a fixed point of T. If $x'_{n+1} \notin K$, then there exists an $x_{n+1} \in \partial K$ such that $d(x_n, x_{n+1})\, d(x_{n+1}, x'_{n+1})$, and $x_n = x_{n+1}$ is impossible.

We shall first show that $\tau_{n+1}, < \max \{\tau_n, \tau_{n-1}\}$.

Case (*a*). Suppose that $x_{n+1}, x_{n+2} \in P$. Then, from (2),

$$\tau_{n+1} = d(x_{n+1}, x_{n+2}) \le b(d(x_n, x_{n+1})) \max \{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} + c(d(x_n, x_{n+1}) \min \{d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\}$$

and $\tau_{n+1} < \tau_n$.

Case (*b*). Suppose that $x_{n+1} \in P, x_{n+2} \in Q$. Then $d(x_{n+1}, x'_{n+2}) = d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x'_{n+2})$, which implies that

$$\begin{aligned} \tau_{n+1} < d(x_{n+1}, x'_{n+2}) &= d(x'_{n+1}, x'_{n+2}) = d(Tx_n\ Tx_{n+1}) \\ &\le b(d(x_n, x_{n+1}) \max \{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\ &\quad + c(d(x_n, x_{n+1})) \min \{d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\} \\ &= b(d(x_n, x_{n+1}) \max \{\tau_n, d(x_{n+1}, x'_{n+2})\}, \end{aligned}$$

which implies that $\tau_{n+1} < \tau_n$.

Case (*c*). Suppose that $x_{n+1} \in Q, x_{n+2} \in P$. Since no two consecutive $x'_n s$ can lie in Q, $x_n \in P$.

Proposition 1. *[1] Let $x, y \in X$, X a Banach space, and $\mu := \lambda x + (1 - \lambda)y$, where $0 \le \lambda \le 1$. Then, for any $w \in X$, $\|w - \mu\| \le \max \{\|w - x\|, \|w - y\|\}$.*

Using Proposition 1, $\tau_{n+1} \le \max \{d(x'_{n+1}, x_{n+2}, d(x_n, x_{n+2})\}$. Suppose that the maximum is $d(x'_{n+1}, x_{n+2})$.

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From (1) it follows that

$$\tau_{n+1} \le d(x_{n+1}, x'_{n+2}) = d(x'_{n+1}, x'_{n+2}) = d(Tx_n, Tx_{n+1})$$
$$\le b(d(x_n, x_{n+1}) \max \{d(x_n Tx_n), d(x_{n+1} Tx_{n+1})\}$$
$$+ c(d(x_n, x_{n+1})) \min \{d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\}$$
$$= b(\tau_n) \max \{d(x_n, x'_{n+1}), \tau_{n+1}\} + c(\tau_n) \min \{d(x_n, x_{n+2}), d(x_{n+1}, x'_{n+1})\}$$
$$\le b(\tau_n) \max \{d(x_n x'_{n+1}), \tau_{n+1}\} + c(\tau_n)\, d(x_n, x_{n+2})$$
$$\le b(\tau_n) \max \{d(x_n, x'_{n+1}), \tau_{n+1}\} + c(\tau_n)\, d(x'n+1, x_{x+2}).$$

If $\max \{d(x_n, x'_{n+1}), \tau_{n+1}\} = d(x_n, x'_{n+1})$, then

$$d(x'_{n+1}, x_{n+1}) \le b(\tau_n)\, d(x_n, x'_{n+1}) + c(\tau_n)\, d(x'_{n+1}, x_{n+2}),$$

or

$$\tau_{n+1} \le \frac{b(\tau_n)}{1 - c(\tau_n)}\, d(x_n, x'_{n+1}) < \tau_{n-1}, \text{ since } x_n \in P, x_{n+1} \in Q.$$

If $\max \{d(x_n, x'_{n+1}, \tau_{n+1}\} = \tau_{n+1}$, then

(3) $$d(x'_{n+1}, x_{n+2}) \le b(\tau_n)\tau_{n+1} + c(\tau_n)\, d(x'_{n+1}, x_{n+2})$$

or

$$\tau_{n+1} \le \frac{b(\tau_n)}{1 - c(\tau_n)}\, \tau_{n+1},$$

a contradiction

Suppose now that $\max \{d(x'_{n+1}, x_{n+2}), d(x_n, x_{n+2})\}$ is $d(x_n, x_{n+2})$. Then

$$\tau_{n+1} \le d(x_n, x_{n+2}) = d(Tx_{n-1}, Tx_{n+1})$$
$$\le b(d(x_{n-1}, x_{n+1}) \max \{d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\}$$
$$c(d(x_{n-1}, x_{n+1})) \min \{d(x_{n-1}, Tx_{n+1}), d(x_{n+1}, Tx_{n-1})\}$$
(4) $$\le b(d(x_{n-1}, x_{n+1})) \max \{\tau_{n-1}, \tau_{n+1}\} + c(d(x_{n-1}, x_{n+1}))\tau_n.$$

If $\max \{\tau_{n-1}, \tau_{n+1}\} = \tau_{n-1}$, then we have

$$\tau_{n+1} \le b(d(x_{n-1}, x_{n+1}))\, \tau_{n-1} + c(d(x_{n-1}, x_{x+1}))\tau_n$$
$$< [b(d(d(x_{n-1}, x_{n+1})) + c(d(x_{n-1}, x_{n+1}))]\tau_{n-1} < \tau_{n-1},$$

since $x_{n-1}, x_n \in P$.

If $\max \{\tau_{n-1}, \tau_{n+1}\} = \tau_{n+1}$, then

$$\tau_{n+1} \le \frac{c(d(x_{n-1}, x_{n+1}))}{1 - b(d(x_{n-1}, x_{n+1}))}\, \tau_n < \tau_{n-1}.$$

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Therefore, in all cases, $\tau_{n+1} < \max\{\tau_n, \tau_{n-1}\}$.

The remaining cases, $x_{n+1}, x_{n+2} \in Q$, cannot occur.

Next we shall show that $\{\tau_n\}$ converges to zero. There are two possibilities. Either $\{x_n\}$ possesses a subsequence $\{x_{n(k)}\}$ with the property that $x_{n(k)+1}, x_{n(k)+2} \in P$, or it doesn't.

Suppose such a sequence exists.

Fact 1. $\tau_{n+1} < max\{\tau_n, \tau_{n-1}\}$ *implies that* $\tau_{n+1} \le max\{\tau_{n-k}, \tau_{n-k-1}\}$ *for each* $0 \le k \le n$.

Proof. The result is trivially true for $k = 0$. Assume the induction hypothesis. Since

$$\tau_{n-k} \le \max\{\tau_{n-k-1}, \tau_{n-k-2}\}$$

$$\tau_{n+1} < \max\{\tau_{n-k}, \tau_{n-k-1}\} < \max\{\max\{\tau_{n-k-1}, \tau_{n-k-2}\}, \tau_{n-k-1}\}$$
$$= \max\{\tau_{n-k-1}, \tau_{n-k-2}\}.$$

It then follows that

(5) $$\tau_{n(k)} \le \max\{\tau_{n(k-1)+1}, \tau_{n(k-1)+2}\}.$$

Consider $\tau_{n(k-1)+1}$. Since $x_{n(k-1)+1}, x_{n(k-1)+2} \in P$, $\tau_{n(k-1)+1} < \tau_{n(k-1)}$. For $\tau_{n(k-1)+2}$, since $x_{n(k-1)+2} \in P$, if $x_{n(k-1)+3} \in P$, then $\tau_{n(k-1)+2} < \tau_{n(k-1)+1}$ by Case (a). If $x_{n(k-1)+3} \in Q$, then $\tau_{n(k-1)+2} < \tau_{n(k-1)+1}$ by Case (b). Therefore $\tau_{n(k)} < \tau_{n(k-1)}$, and $\{\tau_n\}$ converges. Call the limit τ.

Suppose $\tau > 0$. From (5),

(6) $$\tau_{n(k+1)} \le \tau_{n(k)+1} = d(x_{n(k)+1}, x_{n(k)+2}) \le \tau_{n(k)}.$$

Therefore $\lim_k d(x_{n(k)+1}, x_{n(k)+2}) = \tau$.

From Case (a), $d(x_{n(k)+1}, x_{n(k)+2} \le b(\tau_{n(k)})\,\tau_{n(k)}$. Since $\tau_{n(k)} \ge \tau$, and b is decreasing, $\tau_{n(k)+1} \le b(\tau)\tau_{n(k)}$. Taking the limit as $n \to \infty$ yields $\tau \le b(\tau)\tau$, a contradiction. Therefore $\tau = 0$.

Fact 2. *For each* j *sufficiently large there exists* $k = k(j)$ *such that* $n(k) \le j \le n(k+1)$. *Using Fact* 1, $\tau_j \le \tau_{n(k)}$.

Proof. By induction. If $j = n(k)$, then $\tau_j = \tau_{n(k)}$. If $j = n(k)+1$, then $\tau_j < \tau_{j-1} = \tau_{n(k)}$ by Case (a). If $j = n(k)+2$, then $\tau_j < \max\{\tau_{n(k)+1}, \tau_{n(k)}\}$ by what we have just provided.

Suppose that the result is true for $j = n(k)+i$. Then $j+1 \le n(k+1)$. If $j+1 = n(k+1)$, then we have equality. If $j+1 < n(k+1)$, then

$$\tau_{j+1} < \max\{\tau_{n(k)+i}, \tau_{n(k)+i-1}\} \le \max\{\tau_{n(k)}, \tau_{n(k)}\} = \tau_{n(k)},$$

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by the induction hypothesis.

Therefore $\lim \tau_j = 0$.

Now suppose that no such sequence exists. Then, for all n sufficiently large, one must have, for two consecutive values of n, $x_n \in P$ and $x_{n+1} \in Q$..

Let $\{x_{n(i)}\}$ be the subsequence of $\{x_n\}$ consisting of points in Q. Since no two consecutive points can be in Q, $x_{n(i)+1} \in P$. Since no two consecutive points are in P, $x_{n(i)+2} \in Q$; i.e., $n(i)+2 = n(i+1)$. Hence , also, $n(i-1) = n(i) - 2$. Thus the subsequence $\{x_{n(i)}\}$ is either $\{x_{2n}\}$ or $\{x_{2n+1}\}$.

Suppose $\{x_{2n}\} \subset Q$. Then $x_{2n+1} \in P$ and, by Case (c), $\tau_{2n} < \tau_{2n-2}$. Therefore $\{\tau_{2n}\}$ is monotone decreasing and has a limit. Call it τ.

For each n, by Proposition 1,

$$\tau_{2n} = d(x_{2n}, x_{2n+1}) \leq \max \{d(x_{2n-1}, x_{2n+1}), d(x'_{2n}, x_{2n+1})\}.$$

Thus there must exist at least one infinite subsequence of $\{2n\}$ for which either

(7) $$\tau_{2n(h)} \leq d(x_{2n(h)-1}, x_{2n(h)+1})$$

or

(8) $$\tau_{2n(h)} \leq d(x'_{2n(h)}, x_{2n(h)+1}).$$

If (8) is true, then, by Case (c),

(9) $$\tau_{2n(h)} \leq d(x'_{2n(h)+1}) < \tau_{2n(h)-2}.$$

If $\tau > 0$, from the first part of Case (c),

$$\tau_{2n(h)} \leq d(x'_{2n(h)-1}, x'_{2n(h)}) \leq b(\tau_{2n(h)-2})\, \tau_{2n(h)-2}.$$

from Case (b). For all h sufficiently large, $\tau_{2n(h)-1} > \tau/2$. Thus, for each such h,

$$\tau_{2n(h)} \leq b(\tau/2)\, \tau_{2n(h)-1}.$$

Taking the limit as $h \to \infty$ yields $\tau \leq b(\tau/2)\tau$, a contradiction. Therefore $\tau = 0$. By Fact 2, $\lim \tau_j = 0$.

If (7) is satisfied, the, by the second part of Case (c),

(10) $$\tau_{2n(h)} < [b(d(x_{2n(h)-2}, x_{2n(h)}) + c(d(x_{2n(h)-2}, x_{2n(h)})]\tau_{2n(h)-2}.$$

The sequence $\{d(x_{2n(h)-2}, x_{2n(h)})\}$ is bounded. Hence it has a convergent subsequence. Without loss of generality we may assume that $\lim_h (d(x_{2n(h)-2}, x_{2n(h)}) = \rho$. Also, $\{\tau_{2n(h)}\}$ is monotone decreasing, hence convergent. Call the limit τ.

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If $\rho > 0$, choose h so large that $(d(x_{2n(h)-2}, x_{2n(h)}) > \rho/2$. Then taking the limit of (10) as $h \to \infty$ yields

$$\tau \le [b(\rho/2) + c(\rho/2)]\,\tau < \tau,$$

a contradiction. Therefore $\rho = 0$.

Finally,

$$\tau_{2n(h)-2} - d(x_{2n(h)-2}, x_{2n(h)}) \le d(x_{2n(h)-1}, x_{2n(h)}) < d(x_{2n(h)-1}, x'_{2n(h)}) < \tau_{2n(h)-2}.$$

Hence $\lim_h d(x_{2n(h)-1}, x_{2n(h)})$.

Suppose $\{x_{2n+1}\} \subset Q$. Then $x_{2n+1} \in P$ for all n sufficiently large. By Case (c), $\tau_{2n+1} < \tau_{2n-1}$ and $\{\tau_{2n+1}\}$ is monotone decreasing and has a limit $\tau \ge 0$.

For each n, by Proposition 1,

$$\tau_{2n+1} = d(x_{2n+1}, x_{2n+2} \le \max\{d(x_{2n}, x'_{2n+2}), d(x'_{2n+1}, x_{2n+2})\}$$

Then there exists at least one infinite subsequence of $\{2n+1\}$ for which either

(11) $$\tau_{2n(h)+1} \le d(x_{2n(h)}, x_{2n(h)+2})$$

or

(12) $$\tau_{2n(h)+1} \le d(x'_{2n(h)+1}, x_{2n(h)+2}).$$

If (12) holds, then, by Case (c), $\tau_{2n(h)+1} < \tau_{2n(h)-1}$,

and it follows that $\lim \tau_n = 0$.

If (11) holds then, as in case (5), $\lim \tau_n = 0$.

We shall now show that $\{x_n\}$ is Cauchy.

Suppose it is not Cauchy. Then there exist an $\varepsilon > 0$ and two subsequences $\{p(n)\}$ and $\{q(n)\}$ such that, for all n, $p(n) > q(n) \ge n$, $d(x_{p(n)}, x_{q(n)}) \ge \varepsilon$ and $d(x_{p(n)-1}, x_{q(n)}) < \varepsilon$.

Proof. The first inequality is true by the negation of the definition of a Cauchy sequence. The second inequality follows by the following argument. Since $\{\tau_n\}$ is monotone decreasing with limit zero, there exists an N such that $n > N$ implies that $\tau_n < \varepsilon$. Let n be the smallest integer for which $p(n) > q(n) \ge N$. Then $p(n) > q(n) + 1$, for $p(n) = q(n) + 1$ implies that $d(x_{p(n)}, x_{q(n)}) < \varepsilon$, a contradiction. Since $d(x_{p(n)+1}, x_{q(n)}) < \varepsilon$, choose the smallest integer $r > p(n) + 1$ such that $d(x_r, x_{q(n)}) \ge \varepsilon$. Then, with $r := p(n)$, we have $d(x_{p(n)}, x_{q(n)}) \ge \varepsilon$ and $d(x_{p(n)-1}, x_{q(n)}) < \varepsilon$. Now empoly the same argument to $p(n+1)$, $q(n+1)$, etc.We then have, with $S_n := d(x_{p(n)}, x_{q(n)})$,

$$\varepsilon \le s_n \le d(x_{p(n)}, x_{q(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < \tau_{p(n)}) + \varepsilon\ .$$

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Taking the limit as $n \to \infty$, yields $\lim d(x_{p(n)-1}, x_{q(n)}) = \varepsilon$.

Using the triangular inequality,

$$-\tau_{p(n)} - \tau_{q(n)} + x_n \le d(x_{p(n)+1}, x_{q(n)+1}) \le \tau_{p(n)} + \tau_{q(n)} + s_n,$$

$$-\tau_{p(n)} + s_n \le d(x_{p(n)+1}, x_{q(n)}) \le \tau_{p(n)} + s_n,$$

$$-\tau_{q(n)-1} + s_n \le d(x_{p(n),}, x_{q(n)-1}) \le s_n + \tau_{q(n)-1},$$

$$-\tau_{p(n)-1} - \tau_{q(n)-1} + s_n \le d(x_{p(n)-1}, x_{q(n)-1}) \le s_n + \tau_{q(n)-1} + \tau_{p(n)-1}.$$

Taking the limit as $n \to \infty$ in each of the above inequalties yields

$\lim d(x_{p(n)+1}, x_{q(n)+1}) = \lim d(x_{p(n)+1}, x_{q(n)}) = \lim d(x_{p(n)}, x_{q(n)-1})$

$= \lim d\,(x_{p(n)-1}, x_{q(n)-1}) = \varepsilon.$

Also,

$$s_n \le d(x_{p(n)}, x_{q(n)+1}) + \tau_{q(n)} \le s_n + 2\tau_{q(n)}$$

and $\quad s_n \le d(x_{(p(n)-1}, x_{q(n)}) + \tau_{p(n)-1}) \le s_n + 2\tau_{p(n)-1}$

$$= \lim d(x_{p(n)-1}, x_{q(n)}) = \varepsilon.$$

Case (A). Suppose that $x_{p(n)+1}, x_{q(n)+1} \in P$. Then

$$d(x_{p(n)+1}, x_{q(n)+1}) = d(Tx_{p(n)}, Tx_{q(n)})$$

$$\le b(d)x_{p(n)}, x_{q(n)})) \max \{d(x_{p(n),}, Tx_{p(n)}), d(x_{q(n)}, Tx_{q(n)})\}$$

$$+ c(d(x_{p(n)}, x_{q(n)})) \min \{d(x_{p(n)}, Tx_{q(n)}, d(x_{q(n)}, Tx_{p(n)})\}$$

$\le b(\varepsilon) \max \{\tau_{p(n)}, \tau_{q(n)}\} + c(\varepsilon) \min \{d(x_{p(n)}, x_{q(n)+1}), d(x_{q(n),}, x_{p(n)+1})\}.$

Case (B). $x_{p(n)+1} \in P, x_{q(n)+1} \in Q$. Then $x_{q(n)} \in P$, and

$$d(x_{p(n)+1}, x_{q(n)}) = d(Tx_{p(n)}, Tx_{q(n)-1})$$

$$\le b(d(x_{p(n)}, x_{q(n)-1})) \max \{d(x_{p(n)}, Tx_{p(n),}, dx_{p(n)-1}, Tx_{q(n)-1})\}$$

$$+ c(d(x_{p(n)}, x_{q(n)-1})) \min \{d(x_{p(n)}, Tx_{q(n)-1}), d(x_{q(n)-1}), d(x_{q(n)-1}, Tx_{p(n)-1})\}$$

Case (C). $x_{p(n)+1} \in Q, x_{q(n)+1} \in P$.

$$d(x_{p(n)}, x_{q(n)+1}) = d(x_{q(n)-1}, Tx_{q(n)})$$

$$\le b(d(x_{p(n)-1}, x_{q(n)})) \max \{d(x_{p(n)}, x_{q(n),}, dx_{q(n)-1}, Tx_{q(n)-1})\}$$

$$+ c(d(x_{p(n)}, x_{q(n)-1})) \min \{d(x_{p(n)}, Tx_{q(n)-1}, Tx_{p(n)-1})\}$$

Case (D). $x_{p(n)+1}, x_{q(n)+1} \in Q$.

$$d(x_{p(n)}, x_{q(n)}) = d(Tx_{p(n)-1}, Tx_{q(n)-1})$$

$$\le b(d(x_{p(n)-1}, x_{q(n)-1})) \max \{d(x_{p(n)-1,}, x_{\hat{p}(n),}, d(x_{q(n)-1}, x_{q(n)})\}$$

$$+ c(d(x_{p(n)_{-1}}, x_{q(n)-1})) \min \{d(x_{p(n)-1}\, x_{q(n)}), dx_{q(n)-1}, x_{p(n)})\}$$

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Using the facts that

$$\lim d(x_{p(n)}, x_{q(n)-1}) = \lim d(x_{p(n)-1}, x_{q(n)}) = \lim d(x_{p(n)-1}, x_{q(n)-1}) = \varepsilon,$$

we have, for all n sufficiently large, upon adding the inequalities in Cases (a) - (d),

$$\begin{aligned} d(x_{p(n)+1}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{q(n)}) + d(x_{p(n)}, x_{q(n)+1}) + d(x_{p(n)}, x_{q(n)}) \\ \le b(\varepsilon) \max\{\tau_{p(n)}, \tau_{q(n)}\} + c(\varepsilon) \min\{d(x_{p(n)}, x_{q(n)+1}), d(x_{q(n)}, x_{p(n)+1}) \\ + b(\varepsilon/2) \max\{\tau_{p(n)}, \tau_{q(n)}\} + c(\varepsilon/2) \min\{d(x_{p(n)}, x_{q(n)}), d(x_{q(n)-1}, x_{p(n)}) \\ + b(\varepsilon/2) \max\{\tau_{p(n)-1}, \tau_{q(n)}\} + c(\varepsilon/2) \min\{d(x_{p(n)-1}, x_{q(n)+1}), d(x_{q(n)}, x_{p(n)}) \\ + b(\varepsilon/2) \max\{\tau_{p(n)-1}, \tau_{q(n)-1}\} + c(\varepsilon/2) \min\{d(x_{p(n)-1}, x_{q(n)}), d(x_{q(n)-1}, x_{p(n)})\} \end{aligned}$$

Taking the limit as $n \to \infty$ gives

$$4\varepsilon \le c(\varepsilon)\varepsilon + c(\varepsilon/2)\, 3\varepsilon < 4\varepsilon,$$

a contradiction. Therefore $\{x_n\}$ is Cauchy, hence convergent to a point z.

We shall now show that z is a fixed point of T. Since $\tau_n > 0$ for all n, there exists a subsequence $\{x_{h(n)}\}$ of $\{x_n\}$ such that $x_{h(n)} \neq z$. In turn, there must be a subsequence of $\{x_{h(n)}\}$ consisting only of points of P or points of Q. Without loss of generality we shall assume two cases.

Case (E), $\{x_{h(n)}\} \subset Q$, and Case (F), $\{x_{h(n)}\} \subset P$.

For Case (E), $x_{h(n)} \in Q$ implies that $x_{h(n)+1} \in P$. Define $\rho_n := d(z, x_n)$.

$$d(x_{h(n)+1}, Tz) \le b(\rho_{h(n)}) \max\{\tau_{h(n)}, d(z, Tz)\} + c(\rho_{h(n)}) \min\{d(x_{h(n)}, Tz), \rho_{h(n)+1}\}.$$

Fix $\varepsilon > 0$. Then there exists an N such that $n > N$ implies that $\rho_{h(n)} < \epsilon$. Therefore, $c(\rho_{h(n)}) > c(\varepsilon)$, or $-c(\rho_{h(n)}) < -c(\varepsilon)$. Then

$$d(x_{h(n)+1}, T_z) \le (1 - c(\varepsilon)).\ \max\{\tau_{h(n)}, d(z, Tz)\} + \min\{d(x_{h(n)}, Tz), \rho_{h(n)+1}\}.$$

Taking the limit as $n \to \infty$ yields $d(z, Tz) \le (1 - c(\varepsilon))d(z, Tz)$, which implies that $z = Tz$.

For Case (F), we get

$$\begin{aligned} d(x'_{h(n)+1}, Tz) &= d(Tx_{h(n)}, Tz) \\ &\le b(d(x_{h(n)}, z)) \max\{d(x_{h(n)}, x'_{h(n)+1}), d(z, Tz)\} \\ &\quad + c(\rho_{h(n)}, z)) \min\{d(x_{h(n)}, Tz), d(z, x'_{h(n)+1})\}. \end{aligned}$$

Fix $\varepsilon > 0$ and choose n large enough so that $\rho_{h(n)} < \varepsilon$. Then

$$\begin{aligned} (13)\ d(x'_{h(n)+1}, Tz) \le (1 - c(\varepsilon)) \max\{d(x_{h(n)}, x'_{h(n)+1}), d(z, Tz)\} \\ + \min\{d(x_{h(n)}, Tz), d(z, x'_{h(n)+1})\}. \end{aligned}$$

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From Proposition 1, $d(z, x'_{h(n)+1}) \leq \max\{(d, x_{h(n)}, d(z, x_{h(n)+1})\}$. Taking the limit of (13) as $n \to \infty$ yields

$d(z, Tz) \leq (1 - c(\varepsilon))\, d(z, Tz)$,

which implies that $z = Tz$.

The uniqueness of the fixed point follows from the Lemma of [2], since definition (2) is a special case of (24).

Theorem 2. *Let K be a nomempty compact subset of a Banach space X, $T : K \to X$, T continuous and such that $x \in \partial K$ implies $Tx \in K$. Suppose that, for each $x, y \in K$, $x \neq y$,*

$$d(Tx, Ty)\ ,\ b(x, y) \max\{d(x, Tx), d(y, Ty)\} + c(d(x, y) \min\{d(x, Ty), d(y, Tx)\},$$

where b and c are nonnegative decreasing functions from $R^+ \to [0, 1)$ such that $b(t) + c(t) \leq 1$ for all $t > 0$. Then T has a unique fixed point.

Proof. Let $\{x_n\}$ be a general orbit of x_0. By following the first part of the proof of Theorem 1, there are two changes. One is that $\leq$ is replaced every where by strict inequality. The other is that $c(\tau_n) \neq 1$. For, if $c(\tau_n) = 1$, then, from inequality (3), $b(\tau_n) = 0$, and one obtains the contradiction $d(x'_{n+1}, x_{n+2}) < d(x'_{n+1}, x_{n+2})$. In inequality (4) one obtains the inequality $\tau_{n+1} < \tau_{n-1}$.

Since X is compact $\{x_n\}$ has a convergent subsequence $\{x_{n(i)}\}$. Call the limit z.

Then $\{x_{n(i)}\}$ has one of the following three properties :

(P_1) $\{x_{n(i)}\}$ has a subsequence $\{x_{n(k)}\}$ such that $x_{n(k)+1}$ and $x_{n(k)+2} \in P$, or no such subsequence exists. Therefore, for all k sufficiently large, one must have either

(P_2) for all n sufficently large, the subsequence $\{x_{2n}\} \subset Q$, or

(P_3) for all n sufficiently large, the subsequence $\{x_{2n}\} \in P$.

If condition (P_1) is satisfied, then one obtains inequality (6). Taking the limit as $k \to \infty$ yields, using the continuity of T,

(14) $d(z, Tz) = d(Tz, T^2z) = d(z, Tz)$.

If condition (P_2) is satisfied, then $\{\tau_{2n}\}$ is monotone decreasing, hence convergent. If (8) is satisfied, (14) follows from (9) by taking the limit as $h \to \infty$.

If (7) is satisfied, then, by the triangular inequality,

$$\tau_{2n(h)} - d(x_{2n(h)-1}, x_{2n(h)+1}) \leq d(x_{2n(h)-1}, x_{2n(h)}) \leq d(x_{2n(h)-1}, x'_{2n(h)})$$
$$= d(Tx_{2n(h)-2}, Tx_{2n(h)-1}) \leq \tau_{2n(h)-2},$$

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by Case (b). Taking the limit as $h \to \infty$ yields $\lim_h d(x_{2n(h)-1}, x_{2n(h)+1}) = 0$ and $\lim_h d(x_{2n(h)-1}, x'_{2n(h)}) = \lim_h \tau_{2n(h)}$, which implies (14).

The proof for (P_3) is similar.

Suppose that $z \neq Tz$. Then, from (14),

$$d(z, Tz) = d(Tz, T^2z) < b(z, Tz) \max \{d(z, Tz)\, d(Tz, T^2z)\} < d(z, Tz),$$

a contradiction. Therefore. $z = Tz$.

Uniqueness of the fixed point follows in Theorem 1.

Theorems 3.1 and 4.1 of [1] are special cases, respectively, of Theorems 1 and 2 of this paper.

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COMMON FIXED POINT THEOREMS FOR DENSIFYING MAPPINGS

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ABSTRACT

Several fixed point theorems for multivalued compatible maps are proved. As corollaries we obtain several known fixed point theorems for densifying maps.

In a recent paper Diviccaro, Khan and Sessa [4] established the following theorem.

Theorem 1. *Let f and g be two densifying and commuting maps of a bounded complete metric space (X, d) such that*

(1) $d(fx, fy) < \max \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}$

for all x, y in X such that the right hand side of (1) is positive and $fX \subseteq gX$. Then f and g have a unique common fixed point.

The purpose of this note is to prove the aboe result by replacing the hypothesis of commuting by the weaker assumption of compatibility. This goal will be realized by first proving some fixed point theorems for multivalued maps, and then obtaining the desired result as a corollary.

We first state some familiar definitions. For a metric space (X, d), f a selfmap of X, A a bounded subset of X, $\alpha(A)$ denotes the measure of monocompactness of A; i.e., the infimum of all $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameters less than ε.

The following well-known properties of α hold :

(*i*) $0 \leq \alpha(A) \leq \text{diam } A$,

(*ii*) $\alpha(A) = 0$ iff A is precompact,

(*iii*) $\alpha(A \cup B) \max \max \{\alpha(A), \alpha(B)\}$ for any bounded subsets A, B of X,

(*iv*) $A \subseteq B$ implies that $\alpha(A) \leq \alpha(B)$.

f is said to be densifying if f is continuous and, for any bounded non precompact subset A of X, we have $\alpha(fA) < \alpha(A)$.

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Let $B(X)$ denote the set of bounded subsets of a complete metric space (X, d) and define a function $\delta : B(X) \times B(X) \to [0, \infty)$ by $\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}$. It is immediate that δ satisfies the triangular inequality and that $\delta(A, B) = 0$ iff $A = B = \{a\}$. Let I be a selfmap of $X, F : X \to B(X)$. F and I are said to be δ-compatible iff $x \in B(X)$ for $x \in X$ and $\delta(IFx_n, FIx_n) \to 0$ whenever $\{x_n\}$ is a sequence such that $Ix_n \to t$ and $\{Fx_n\} \to \{t\}$ for some t in X. (See, e.g., Jungck and Rhoades [6]).

Theorem 2. *Let F be a continuous mapping of a complete metric space (X, d) into $B(X)$, I a continuous selfmap of X Such that*

(2) $\delta(Fx, Fy) \le c \max \{d(Ix, Iy), \delta(Ix, Fx), \delta(Iy, Fy), \delta(Ix, Fy), \delta(Iy, Fx)\}$

for all x, y in X, where $0 \le c \le 1$. If $FX \subseteq IX$ and F and I are δ-compatible, then F and I have a unique common fixed point z and further $Fz = \{z\}$.

Proof. The proof follows along the lines of the proof of theorem 1 of Fisher [5]. One defines a sequence $\{x_n\}$ by $x_0 \in X, y_1 \in Fx_0$. Since $FX \subseteq IX$, choose x_1 so that $Ix_1 = y_1$. In general x_n is chosen so that $Ix_n = y_n, y_n \in Fx_{n-1}$. Then, as in Fisher, it follows that Ix_n converges to a point z, and the sets $\{Fx_n\}$ converge to the set $\{z\}$.

From (2),

$$\begin{aligned}\delta(FIx_n, Fx_n) &\le c \max \{\delta(I^2x_n, Ix_n), \delta(I^2x_n, FIx_n), \delta(Ix_n, Fx_n),\\ &\qquad \delta(I^2x_n, Fx_n), \delta(Ix_n, FIx_n)\}\\ &\le c \max \{\delta(IFx_{n-1}, Ix_n), \delta(IFx_{n-1}, FIx_n)\, \delta(Ix_n, Fx_n),\\ &\qquad \delta(IFx_{n-1}, Fx_n), \delta(Ix_n, FIx_n)\}. \qquad (3)\end{aligned}$$

From the definition of δ-compatibility, $\delta(IFx_{n-1}, FIx_n) \to 0$. Since F and I are continuous, it then follows that $\lim \delta(IFx_{n-1}, Ix_n) = \lim \delta(FIx_n, Ix_n) = \delta(Fz, z)$ and $\lim \delta(IFx_{n-1}, Fx_n) = \lim \delta(FIx_n, Fx_n) = \delta(Fz, z)$. Taking the limit of (3) as $n \to \infty$, we obtain $\delta(Fz, z) \le c \max \{\delta(Fz, z), \delta(Fz, Fz)\} = \delta(Fz, Fz)$. Again using (2),

$$\delta(FIx_n, FIx_n) \le c \max \{d(I^2x_n, I^2x_n), \delta(I^2x_n, FIx_n)\}.$$

Taking the limit as $n \to \infty$ yields $\delta(Fz, Fz) \le c\delta(Fz, Fz)$, from which it follows that Fz is the singleton set $\{z\}$.

Since $FX \subseteq IX$, there exists a point w such that $Iw = z$. Using (2),

$$\delta(Fx_n, Fw) \le c \max \{d(Ix_n, Iw), \delta(Ix_n, Fx_n), \delta(Iw, Fw), \delta(Ix_n, Fw), \delta(Iw, Fx_n)\}.$$

Taking the limit as $n \to \infty$ gives $\delta(z, Fw) \le c\delta(z, Fw)$, which implies that $Fz = \{z\} = Iw$. From the δ-compatibility of I and F it follows that

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$\{z\} = Fz = FIw = IFw = \{Iz\}$, and z is a fixed point of I. Condition (2) forces the common fixed point to be unique.

Theorem 3. *Let F be a continuous mapping of a compact metric space (X, d) into $B(X)$, I a continuous selfmap of X such that*

(4) $\delta(Fx, Fy)\,\delta(Fx, Fy) < \max\{d(Ix, Iy), \delta(Ix, Fx), \delta(Iy, Fy), \delta(Ix, Fy), \delta(Iy, Fx)\}$

for all x, y in X for which the right-had-side of (4) is positive. If $FX \subseteq IX$ and F and I are δ-compatible, then F and I have a unique common fixed point z and further $Fz = \{z\}$.

Proof. Suppose there exists a $0 \le c < 1$ such that F and I satisfy inequality (2) for all x, y in X for which the right had side of inequality (2) is positive. If the right-hand-side of (2) is zero it follows that we must have $Fx = Fy = \{Ix\} = \{Iy\}$, which forces the left had side of (2) to be zero, so (2) is satisfied for all x, y in X and the result follows from Theorem 2.

If no such number c exists, then there is an increasing sequence $\{c_n\}$, with limit 1, and sequences $\{x_n\}$, $\{y_n\}$ such that

$$d(fx_n, Fy_n) \ge c_n \max\{d(Ix_1, Iy_n), \delta(Ix_n, Fx_n), \delta(Iy_n, Fy_n), \delta(Ix_n, Fy_n), \delta(Iy_n, Fx_n)\}.$$

Since X is compact, we may assume, without loss of generality, that the sequecnes $\{x_n\}$ and $\{y_n\}$ converge to points x and y respectively. Letterring n tend to infinity in the above inequality leads to, since F and I are continuous,

$$\delta(Fx, Fy) \ge \max\{d(Ix, Iy), \delta(Ix, Fx), \delta(Iy, Fy), \delta(Ix, Fy), \delta(Iy, Fx)\}:$$

Using the above inequality, along with (4) it follows that we must have $Fx = Fy = \{Ix\} = \{Iy\}$. From the δ-compatibility of I and F, we then have $FIx = IFx$. Thus $F^2x = FIx = FIx = IFx = \{I^2x\}$.

Suppose that $FIx \ne Fx$. Then, from (4),

$$\delta(FIx, Fx) < \max\{d(I^2x, Ix), \delta(I^2x, FIx), \delta(Ix, Fx), \delta(I^2x, Fx), \delta(Ix, FIx)\}$$
$$= \delta(FIx, Fx),$$

a contradiction. Therefore $FIx = Fx$, and Ix is a fixed point of F. With $Ix = z$ it follows that $\{Iz\} = \{I^2x\} = FIx = Ix = z$, and z is also a fixed point of I. Definition (4) forces uniqueness of the common fixed point.

Corollary 1. *Let f and g be continuous selfmaps of a bounded metric space (X, d) satisfying inequality (1) for all x,y in X for which the right-hand-side of (1) is positive. If $fX \subseteq gX$ and f and g are compatible, then f and g have a unique common fixed point z.*

Proof. Define $A = O(x_0)$, the orbit of x, for some x_0 in X. Since X is bounded, so is A. Moreover, $A = \{x_0\} \cup \{f(A)\} \cup \{g(A)\}$. Thus $\alpha(A) = \max\{f(A), \alpha(g(A))\}$. Since f and g are densifying and X is complete, it follows that A is compact. Now apply Theorem 3.

Corollary 2. *Let (X, d) be a complete metric space, f a densifying selfmap of X satisfying*

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$$d(fx, fy) < \max \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$$

for each x, y *in* $X, x \neq y$. *If, for some* x_0 *in* X, *the sequence* $\{x_n\}$ *defined by* $\{x_0, x_1 = f(x_0), x_2 = f(x_1), \ldots\}$ *is bounded, then* f *has a unique fixed point.*

Proof. With $A = O(x)$, A is bounded, and the result follows from Corollary 1 by setting $g = f$.

Theorem 4. *Let* f, g *be selfmaps of a complete metric space* (X, d), f *possessing a unique fixed point* z *and* g *commuting with* f. *Then* z *is the unique common fixed point of* f *and* g.

Proof. $gz = gfz = fgz$, and gz is also a fixed point of f. Since the fixed point of f is unique, $gz = z$.. Suppose that w is also a common fixed point of f and g. Then w is a fixed point of f. Since f has a unique fixed point, $w = z$.

Remarks. 1. Theorems 1, 2, and Corollary 2 of Fisher [5] are special cases of Theorems 2,3, and Corollary 1, respectively, of this paper.

2. Theorem 4 of Diviccaro, Khan and Sessa [4] is a special case of Corollary 1.

3. Theorem 1 of Achari [1], Theorem 3 of Chatterjee [2], and Theorem 1 of Chattopadhyay [3] are special cases of Corollary 2.

4. Theorem 2 of Achari [1] is a special case of Theorem 4.

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SOME REMARKS ON A FIXED POINT IN A-METRIC SPACE

by

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ABSTRACT

In this paper, we follow Rawat and Shahu, [3] and generalise the result using Patil and Achari [2], and Dubey [1] for pair of mappings in 2-metric space.

Theorem 1 :

Let f, g be a pair of self maps of a complete metric space X. If there exist positive real numbers $0 \le \beta\ \gamma, \delta < 1$ *satisfying* $\alpha + \beta < 1$, $\beta + \delta < 1$, and if (1) Let

$$(1)\ (d(fu_1, gu_2))^2 \le \alpha\, d(u_1, fu_3)\, d(u_2, gu_4) + \beta d(u_1, u_2)^2 + \gamma d(u_2, gu_4)\, d(u_3, gu_6) + \delta d(u_4, fu_5)^2$$

for all u_i, $i = 1, 2, \ldots, 6 \in X$ then f, g have a common unique fixed point.

Proof : Let $x, y \in X$ and points

$$u_1 = gfx,\ u_2 = fgy,\ u_3 = gy,\ u_4 = fx,\ u_5 = x,\ u_6 = y$$

Then

$$(d(fgfx, gfgy))^2 \le (\alpha + \beta)\, (d(gfx, fgy))^2$$

if $\alpha + \beta < 1 = k^2$, then $0 < k < 1$

$$d(fgfx, gfgy) \le d(gfx, fgy$$

If we put $x_n = fx_{n-1}, x_{n+1} = gx_n$ for every $n = 1, 2, 3$

and $x = x_{n-3},\ y = x_{n-2}$, then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fgf\, x_{n-3}, gfg\, x_{n-2}) \\ &\le kd(Ifx_{n-3}, fgx_{n-2}) \\ &\le kd(x_{n-1}, x_n) \\ &\le k^2 d(x_{n-2}, x_{n-1}) \\ &\ \ \ldots \quad \ldots \quad \ldots \\ &\le K^n d(x_0, x_1) \end{aligned}$$

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where $K < 1$. Therefore $K^n \to 0$ as $n \to \infty$.

So the sequence defined above is Cauchy sequence in a complete metric space X. There exists a point z in X such that $x_n \to z$, $n \to \infty$

Let $u_1 = u_3 = u_5 = z$, $u_2 = u_4 = u_6 = x_n$

$$(d(fz, x_{n+1}))^2 = (d(fz, gx_n))^2 \le \alpha d(z, fz)\, d(x_n, gx_n) + \beta d(z, x_n)^2 + \gamma d(x_n, gx_n) d(z, gx_n) + \delta d(x_n, fz)^2$$

$n \to \infty$

$$(d(fz, z))^2 \le \alpha d(z, fz) d(x_n, x_{n+1}) + \beta d(z, x_n)^2 + \gamma d(x_n, x_{n+1}) d(z, x_{n+1}) + \delta d(x_n, fz)^2,$$

$n \to \infty$

$$(d(fz, z))^2 \le \alpha.0 + \beta.0 + \gamma.0 + \delta d(z, fz)^2,$$

$$(d(fz, z))^2 \le \delta (d(z, fz))^2.$$

A contradiction $0 < \delta < 1$.

Hence $fz = z$ i.e. z is a fixed point of f. Similarly z is also a fixed of g.

To claim the uniqueness we say that

$$(d(z, w))^2 = (d(fz, gw))^2$$

$$\le \alpha d(z, fz) d(w, gw) + \beta d(z, w)^2 + \gamma d(w, gw)\, d(z, gw) + \delta d(w, fz)^2$$

$$d(z, w)^2 \le (\beta + \delta)\, d(z, w)^2$$

$$\beta + \delta < 1,$$

which is a contradiction. Hence $z = w$.

Theorem 2 :

If $f = g$, we get, which we give without proof.

$$(d(fu_1, fu_2))^2 \le \alpha\, d(u_1, fu_3) d(u_2, fu_4) + \beta d(u_1, u_2)^2 + \gamma d(u_2, fu_4) d(u_3, fu_6) + \delta d(u_4, fu_5)^2$$

$0 < u_1, u_2, \ldots, u_6 \in X < 1$ then f has a fixed point. We state and prove the following theorem which is direct application of the theorem 1 for a family of mappings.

Theorem 3 :

Let $\{f_k\}$ $(k = 1, 2, 3, \ldots, n)$ be a family of mapping of a complete metric space X into itself. If f_k satisfies the conditions

(*i*) $f_1 f_2 \cdots f_n$. commute with every f_k.

(*ii*) $(d(f_1 f_2 \ldots f_n u_1, f_{n-1} \ldots f_2 f_1 u_2))^2$

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$$\leq \alpha d(u_1, f_1 f_2 \cdots f_n u_3) d(u_2, f_n f_{n-1} \cdots f_2 f_1 u_4) + \beta d(u_1, u_2)^2$$
$$+ \gamma d(u_2, f_n f_{n-1} \cdots f_2 f_1 u_4)\, d(u_3, f_n f_{n-1} \cdots f_1 u_6)$$
$$+ \delta d(u_4, f_1 f_2 \cdots f_n u_5)^2$$

$u_1 \ldots u_6$ in X and $0 \leq \alpha, \beta \ldots < 1$ then $\{f_k\}$ have a common fixed point.

Proof :

From (*ii*) If we put $f_1 \ldots f_n = f, \quad f_n \ldots f_1 = g$

Then it takes the form (1) by theorem 1 f, g have a unique fixed point of z. *i.e.* $fz = z = gz$ for any $f_k, f_k(z) = f_k z$ by a view of (1)

$f(k, k\, z) = f_k\, z$ so $f_k(z)$ is fixed point of f and z is a fixed point of z. By putting $u_1 = u_3 = u_5 = fkz,\ u_2 = u_4 = u_6 = z$ in (1)

$$d(f_k z, z)^2 = d(ff_k z, gz)^2$$

$$\leq \alpha d(f_k z, ff_k z)\, d(z, gz) + \beta d(f_k z, z)^2 + \gamma d(z, gz)\, d(f_k\, z, gz) + \delta d(z, ff_k\, z)^2$$

$$d(f_k z, z)^2 \leq (\beta + \delta) d\,(z, f_k z)^2,\ \ (\beta + \delta < 1).$$

$$d(f_k z, z)^2 \leq (\beta + \delta)\, d(f_k z, z)^2.$$

A contradiction. Hence $fkz = z$ i.e. z is a fixed point of a family of mappings $\{f_k\}$.

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RAYLEIGH-TAYLOR INSTABILITY OF ROTATING OLDROYDIAN VISCOELASTIC FLUIDS IN POROUS MEDIUM IN PRESENCE OF A VARIABLE MAGNETIC FIELD

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ABSTRACT

The Rayleigh-Taylor instability of Oldroydian viscoelastic fluid in porous medium in the presence of uniform rotation and variable magnetic field is considered. The magnetic field, the viscosity and the density are assumed to be exponentially varying. For stable density stratification, the system is found to be stable for distrubances of all wave numbers. The magnetic field stablizes the potentially unstable stratification for small wave- length perturbations which are otherwise unstable. The long wave length perturbations remain unstable and are not stablized by magnetic field. Rotation does not affect the stability or instability, as such, of a stratification.

1. Introduction

A detailed account of the instability of the plane interface between two Newtonian fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar [1]. Bhatia [2] has considered the Rayleigh-Taylor instability of two viscous superposed conducting fluids in the presence of a uniform horizontal magnetic field. Sharma [3] has studied the instability of the plane interface between two oldroydian viscoelastic superposed conducting fluids in the presece of a uniform magnetic field. Bhatia and Steiner [4] have studied the problem of thermal instability of a Maxwellian viscoelastic fluid in the presence of rotation and have found that the rotation has a destabilizing effect in contrast to the stabilizing effect on Newtonian fluid. Bhatia and Steiner [5] have also considered the problem of thermal instability of a Maxwellian fluid in hydromagnetics and have found that the magnetic field has stabilizing effect on viscoelastic fluid just as in the case of Newtonian fluid. Eltayeb [6] has stuied the convective instability in a rapidly rotating oldroydian viscoelastic fluid. The medium has been considered to be non-porous in all the above studies.

When the fluid slowly percolates through the pores of a macroscopically homogeneous and isotropic porous medium, the gross effect

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is represented by Darcy's law according to which the usual viscous term in the equations of fluid motion is replaced by the resistance term $-(\mu/k_1)\vec{v}$, where μ is the viscosity of the fluid, k_1 the permeability of the medium and $\vec{v}$ the filter velocity of the fluid. Lapwood [7] has studied the sability of convective flow in hydromagnetics in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through porous medium has been considered by Wooding [8]. Oldroyd [9] proposed a theoretical model for a class of viscoelastic fluids. An experimental demonstration by Toms and Strawbridge [10] reveals that a dilute solution of methyl methacrylate in *n*-butyl acetate agrees well with the theoretical model of Oldroyd fluid.

The present paper attempts to study the stability of the plane interface separating two incompressible superposed rotating Oldroydian viscoelastic fluids in porous medium in presence of a variable magnetic field. The instability of such viscoelastic fluids in porous medium may find applications in geophysics.

2. Perturbation Equations

Let $T_{ij}, z_{ij}, e_{ij}, \mu, \lambda, \lambda_0 (< \lambda), p, \delta_{ij}, v_i, x_i$ and $\frac{d}{dt}$ denote respectively the total stress tensor, the shear stress tensor, the rate-of-strain tensor, the viscosity, the stress relaxation time, the strain retardation time, the isotropic pressure, the Kronecker delta, the velocity vector, the position vector and the mobile operator. Then the Oldroydian viscoelastic fluid is described by the constitutive relations

$$T_{ij} = -p\delta_{ij} + z_{ij},$$

$$(i + \lambda\frac{d}{dt})\, z_{ij} = 2\mu(1 + \lambda_0\frac{d}{dt})e_{ij}, \qquad \ldots (1)$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Relations of the type (1) were proposed and studied by Oldroyd [9]. Oldroyd [9] also showed that many rheological equations of state of general validity, reduce to (1) when linearized. $\lambda_0 = 0$ yields the fluid to be Maxwellian whereas $\lambda = \lambda_0 = 0$ gives the Newtonian viscous fluid.

Consider a static state in which an incompressible Oldroydian viscoelastic fluid is arranged in horizontal strata in porous medium and the pressure p and the density ρ are funtions of the vertical coordinate z only. The system is acted on by a variable horizontal magnetic field $\vec{H}(H_0(z), 0, 0)$, a uniform rotation $\vec{\Omega}(0, 0, \Omega)$ and a gravity force $\vec{g}(0, 0, -g)$. The character of the equilibrium of this initial static state is determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\vec{v}(u, v, w)$, $\delta\rho$, δp and $\vec{h}(h_x, h_y, h_z)$ denote respectively the perturbations in fluid velocity $(0, 0, 0)$, fluid density ρ, fluid pressure p

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and the magnetic field $\vec{H}(H_0(z), 0, 0)$. Then the linearized hydromagnetic perturbation equations of rotating Odroydian viscoelastic fluid in porous medium are

$$\frac{\rho}{\epsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial\vec{v}}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\nabla\,\delta p+\vec{g}\,\delta\rho+\frac{2\rho}{\epsilon}(\vec{v}+\vec{\Omega})\right.$$
$$\left.+\frac{\mu_e}{4\pi}\{(\nabla\times\vec{h})\times\vec{H}+(\nabla\times\vec{H})\times\vec{h}\}\right]-\frac{\mu}{k_1}(1+\lambda_0\frac{\partial}{\partial t})\vec{v}, \quad \ldots (2)$$

$$\nabla\,.\,\vec{v}=0, \quad \ldots (3)$$

$$\epsilon\frac{\partial\vec{h}}{\partial t}=(\vec{H}\,.\,\nabla)\vec{v}-(\vec{v}\,.\,\nabla)\vec{H}, \quad \ldots (4)$$

$$\nabla\,.\,\vec{h}=0, \quad \ldots (5)$$

$$\epsilon\frac{\partial}{\partial t}\delta\rho=-w(D\rho), \quad \ldots (6)$$

where ϵ is the medium porosity, μ_e the magnetic permeability and $D=d/dz$. Equation (6) results from the fact that the density of every particle remains unchanged as we follow it with its motion.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp(ik_x x+ik_y y+nt), \quad \ldots (7)$$

where k_x, k_y are horizontal wave numbers, $k^2=k_x^2+k_y^2$ and n is a complex constant.

For perturbations of the form (7), Eqs. (2)-(6) give

$$\frac{\rho}{\epsilon}(1+\lambda n)nu=-(1+\lambda n)ik_x\,\delta p+(1+\lambda n)\frac{\mu_c}{4\pi}h_z(DH)$$
$$+(1+\lambda n)\frac{2\rho\Omega}{\epsilon}v-\frac{\mu}{k_1}(1+\lambda_0 n)u, \quad (8)$$

$$\frac{\rho}{\epsilon}(1+\lambda n)nv=-(1+\lambda n)ik_y\,\delta p+(1+\lambda n)\frac{\mu_c H}{4\pi}(ik_x h_y-ik_y h_x)$$
$$-(1+\lambda n)\frac{2\rho\Omega}{\epsilon}u-\frac{\mu}{k_1}(1+\lambda_0 n)v, \quad (9)$$

$$\frac{\rho}{\epsilon}(1+\lambda n)nw=-(1+\lambda n)\,[D\delta p+g\delta\rho]+(1+\lambda n)\frac{\mu_e H}{4\pi}\cdot$$
$$[(ik_x h_z-Dh_x)-h_x\frac{DH}{H}]-\frac{\mu}{k_1}(1+\lambda_0 n)w, \quad \ldots (10)$$

$$ik_x u+ik_y v+Dw=0, \quad \ldots (11)$$

$$ik_x h_x+ik_y h_y+Dh_z=0, \quad \ldots (12)$$

$$\epsilon n h_x=ik_x Hu-w(DH), \quad \ldots (13)$$

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$$\in nh_y = ik_x Hv, \qquad \ldots (14)$$

$$\in nh_z = ik_x Hw, \qquad \ldots (15)$$

$$\in n\,\delta\rho = -\,w\,D\rho. \qquad \ldots (16)$$

Multiplying Eqs. (8) and (9) by $by - ik_x, -ik_y$ respectively, adding and using Eqs. (11), (13)-15, we obtain

$$\frac{\rho}{\in} n(1+\lambda n)Dw = -\,k^2(1+\lambda n)\delta p - \frac{\mu}{k_1}(1+\lambda_0 n)\,Dw$$

$$-\frac{2\rho\Omega}{\in}(1+\lambda n)\zeta + (1+\lambda n)k_x k_y \frac{\mu_e H^2}{4\pi n\in}\zeta + (1+\lambda n)\frac{\mu_e H k^2}{4\pi H\in}\,w(DH), \qquad \ldots (17)$$

where ζ the z-component of vorticity, is given by

$$\zeta \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = ik_x v - ik_y\,u.$$

Multipying Eqs. (8) and (9) by $-ik_y$ and $+ik_x$, respectively, adding and using Eqs. (11), (13)-(15), we obtain

$$\zeta = \frac{2(1+\lambda n)\,\Omega\,Dw}{(1+\lambda n)n + (1+\lambda_0 n)\dfrac{\in\nu}{k_1} + (11+\lambda n)\dfrac{k^2 x V_A^2}{n}}, \qquad \ldots (18)$$

where $V_A^2 = \mu_e H^2/4\pi\rho$ is square of the Alfven velocity and $\nu(=\mu/\rho)$ stands for kinematic viscosity. Eliminating δp between Eqs. (10) and (17), using (18) and the relation

$$ik^2 u = -(k_x Dw + k_y \zeta) = -\left[k_x + \frac{2(1+\lambda n)k_y\,\Omega\,n}{(1+\lambda n)n^2 + (1+\lambda_0 n)n\dfrac{\in\nu}{k_1}} + (1+\lambda n)k_x^2 V_A^2\right] Dw, \qquad \ldots (19)$$

we get after simplification

$$[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in\nu}{k_i}][D(\rho Dw) - k^2\rho w] + [gk^2(\frac{1+\lambda n)}{n}]\,(D\rho)w$$

$$+ 4(1+\lambda n)^2\,\Omega^2 n\left[\frac{\rho Dw}{(1+\lambda n)n^2 + (1+\lambda_0 n)n\dfrac{\in\nu}{k_1} + (1+\lambda n)k_x^2 v_A^2}\right]$$

$$+ (1+\lambda n)\frac{\mu_e k_x^2}{4\pi n}\,[D(H^2 Dw) - k^2 H^2 w] = 0. \qquad \ldots (20)$$

3. The Case of Exponentially Varying Density, Viscosity and Magnetic Field

Assume the stratifications in density, viscosity and Assume the stratifications in density, viscosity and magnetic field of the form

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$$\rho = \rho_0 e^{\beta z},\ \mu = \mu_0 e^{\beta z},\ H^2 = H^2 = H_0^z\, e^{\beta z}, \qquad \ldots (21)$$

where ρ_0, μ_0, H_0 and β are constants. Equations (21) imply that the coefficient of kinematic viscosity ν and the Alfven velocity are constant every where. Using the stratification of the form (21), Eq. (20) trnasforms to

$$\left[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in \nu_0}{k_1} + \frac{(1+\lambda_n)}{n} V_A^2 k_x^2 + \frac{4(1+\lambda n)^2 \Omega^2 n}{(1+\lambda n)n^2 + (1+\lambda_0 n)n\frac{\in \nu_0}{k_1} + (1+\lambda n)V_A^2 k_x^2}\right] D^2 w$$

$$+\left[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in \nu_0}{k_1} + \frac{(1+\lambda n)}{n} V_A^2 k_x^2 + \frac{4(1+\lambda n)^2 \Omega^2 n}{(1+\lambda n)ns + (1+\lambda_0 n\frac{E\nu_0}{k_1} + (1+\lambda n)V_A^2 k_x^2}\right] \beta D w \qquad \ldots (22)$$

$$-\left[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in \nu_0}{k_1} + \frac{(1+\lambda n)}{n} V_A^2 k_x^2 - g\beta\frac{(1+\lambda n)}{n}\right] k^2 w = 0,$$

where $\nu_0 = \dfrac{\mu_0}{\rho_0}$ and $V_A^2 = \dfrac{\mu_e H_0^2}{4\pi\rho_0}$ are constants.

The general solution of Eq. (22) is

$$w = A_1 e^{q_1 z} + A_z e^{q_{2z}}, \qquad \ldots (23)$$

where A_1, A_2 are two arbitrary constants and q_1, q_2 are the roots of the equation

$$\left[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in \nu_0}{k_1} + \frac{(1+\lambda n)}{n} V_A^2 k_x^2 + \frac{4(1+\lambda n)^2 \Omega^2 n}{(1+\lambda n)n^2 + (1+\lambda_0 n\frac{\in \nu_0}{k_1} + (1+\lambda n)V_A^2 k_x^2}\right] q^2$$

$$+\left[(1+\lambda n)n + (1+\lambda_0 n)\frac{\in \nu_0}{k_1} + \frac{(1+\lambda n)}{n} V_A^2 k_x^2 + \frac{4(1+\lambda n)^2 \Omega^2 n}{(1+\lambda n)ns + (1+\lambda_0 n\frac{\in \nu_0}{k_1} + (1+\lambda n)V_A^2 k_x^2}\right] \beta q$$

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$$-\left[(1+\lambda n)n+(1+\lambda_0 n)\frac{\in \nu_0}{k_1}+\frac{(1+\lambda n)}{n}V_A^2k_x^2-g\beta\frac{(1+\lambda n)}{n}\right]k^2=0. \quad \ldots (24)$$

If the fluid is supposed to be confined between two rigid planes at $z=0$ and $z=d$, then the vanishing of w at $z=0$ is satisfied by the choice

$$w=A\,(e^{q_1 z}-e^{q_2 z}), \quad \ldots (25)$$

while the vanishing of w at $z=d$ requires

$$\exp(q_1-q_2)d=1, \quad \ldots (26)$$

which imply that

$$(q_1-q_2)d=2im\pi, \quad \ldots (27)$$

where m is an integer.

Eq. (24) gives

$$q_{1,2}=\left[-\beta/2\left\{(1+\lambda n)n+(1+\lambda_0 n)\frac{\in \nu_0}{k_1}+\frac{(1+\lambda n)}{n}V_A^2k_x^2+\frac{4(1+\lambda n)^2\Omega^2 n}{(1+\lambda n)n^2+(1+\lambda_0 n)n\frac{\in \nu_0}{k_1}+(1+\lambda n)V_A^2k_x^2}\right\}\right.$$

$$\pm\frac{1}{2}\left[betasup2\left\{(1+\lambda n)n+(1+\lambda n)\frac{\in \nu_0}{k_1}+\frac{(1+\lambda n)}{n}V_A^2k_x^2+\frac{4(1+\lambda n)^2\Omega^2 n}{(1+\lambda n)n^2+(1+\lambda_0 n\frac{\in \nu_0}{k_1}+(1+\lambda n)V_A^2k_x^2}\right\}^2\right.$$

$$\left.+4k^2\left\{(1+\lambda n)n+(1+\lambda_0 n)\frac{\in \nu_0}{k_1}+\frac{(1+\lambda n)}{n}V_A^2k_x^2+\frac{4(1+\lambda n)^2\Omega^2 n}{(1+\lambda n)n^2+(1+\lambda_0 n\frac{\in \nu_0}{k_1}+(1+\lambda n)V_A^2k_x^2}\right\}\right]^{1/2}\Bigg].$$

$$\left[(1+\lambda n)n+(1+\lambda_0 n)\frac{\in \nu_0}{k_1}+\frac{(1+\lambda_n)}{n}V_A^2k_x^2\right.$$

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$$+\frac{4(1+\lambda n)^2\Omega^2 n}{(1+\lambda n)n^2+(1+\lambda_0 n\frac{\epsilon\nu_0}{k_1}+(1+\lambda n)V_A^2k_x^2}\Bigg]^{-1} \quad \ldots (28)$$

Inserting the values of q_1, q_2 from (28) in Eq. (27) and simplifying, we obtain

$$A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0=0, \quad \ldots (29)$$

where $A_6=A\lambda^2, \qquad A_5=2A\lambda[1+\lambda_0],$

$$A_4=[A\{1+\lambda_0^2\frac{\epsilon^2\nu_0}{k_1^2}+2(\lambda+\lambda_0)\}+B\{\lambda^2\Omega^2\}+\lambda^2\{2k_x^2v_A^2v_A^2A-c\}],$$

$$A_3=\left[2A\{1+\lambda_0\frac{\epsilon^2\nu_0^2}{k_1^2}\}+2B\{\lambda\Omega^2\}+\lambda\{(2+\lambda_0\frac{\epsilon\nu_0}{k_1})(2k_x^2V_A^2A-C)\right],$$

$$A_2=\left[A\left\{\frac{\epsilon^2\nu_0^2}{k_1^2}\right\}+B\{\Omega^2\}+(2k_x^2V_A^2A-C)\left\{1+(\lambda+\lambda_0)\frac{\epsilon\nu_0}{k_1}\right\}+\lambda^2k_x^2V_A^2(k_x^2V_A^2A-C)\right]$$

$$A_1=2\lambda k_x^2V_A^2(k_x^2V_A^2A-C)+\frac{\epsilon\nu_0}{k_1}(2k_x^2v_A^2A-C),$$

$$A_0=k_x^2V_A^2\,(k_x^2V_A^2\,A-C), \quad \ldots (30)$$

where, we have put

$$A=\beta^2d^2+4m^2\pi^2+4k^2d^2,$$

$$B=\beta^2d^2+4m^2\pi^2,$$

$$C=4k^2d^2g\beta.$$

Eq. (29) is the dispersion relation studying the effect of rotation and the variable (exponential) horizontal magnetic field on the stability of stratified (exponentially varying density, viscosity) oldroydian viscoelastic field in porous medium.

For stable stratification $\beta<0$, Eq. (29) does not have any change of sign and so has no positve root of n and the system is always stable for disturbances of all wave numbers.

For unstable stratification and if

$$V_A^2<\frac{4g\beta k^2}{(\beta^2+4m^2\frac{\pi^2}{d^2}+4k^2)kx^2}, \quad \ldots (31)$$

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the constant term in Eq. (29) is negative, therefore, it has at least one positive real root and hence, the system is unstable for all wave numbers satisfying the inequality

$$k^2 < \frac{g\beta \sec^2 \theta}{V_A^2} - \frac{\beta^2 d^2 + 4m^2\pi^2}{4d^2}, \qquad \ldots (32)$$

where θ is the angle between k_x and k. (i.e. $k_x = k \cos \theta$).

If $\beta > 0$ (unstable stratification) and also

$$V_A^2 > \frac{4\beta k^2}{\left(\beta^2 + \frac{4m^2\pi^2}{d^2} + 4k^2\right)k_x^2}, \qquad \ldots (33)$$

then Eq. (29) has no positive root and so the system is stable.

Thus, for unstable density stratification and magnetic field such that

$$V_A^2 > \frac{4\beta k^2}{\left(\beta^2 + \frac{4m^2\pi^2}{d^2} + 4k^2\right)k_x^2}, \qquad \ldots (34)$$

the system is unstable for all wave unmbers satisfying

$$k^2 < g\beta \frac{\sec^2\theta}{V_A^2} - \frac{\beta^2 d^2 + 4m^2\pi^2}{4d^2}.$$

Also, it is clear from Eq. (29) that rotation does not affect the stability or instability , as such, of a stratification.

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FREE CONVECTIVE MHD FLOW OF A DUSTY VISCOELASTIC LIQUID IN POROUS MEDIUM PAST AN OSCILLATING INFINITE POROUS PLATE

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ABSTRACT

Free convective laminar flow of an electrically conducting incompressible dusty viscoelastic liquid (Walters liquid model-*B*) embedded with non-conducting dust particles through a porous medium past an infinite porous plate subject to a slightly sinusoidal transverse velocity distribution has been analysed. The mathematical analysis is presented for the hydromagnetic boundary layer flow without taking into account the induced magnetic field. Expressions for velocity field, temperature distribution, skin- friction and heat transfer have been obtained and discussed with the help of graphs.

1. INTRODUCTION. The study of the flow through porous media is of principal interest due to its applications in geophysics in the assessment of geothermal resources, in design of under ground energy storage systems, oil recovery, soil sciences, astrophysics, nuclear power reactor engineering and so on. Lighthill [1] has studied laminar skin-friction and heat transfer to fluctuating in stream velocity. Stability laminar flow of a dusty gas has been studied by Saffman [5]. Mukherjee and Mukherjee [3] have investigated a problem on unsteady flow of dusty viscous fluid due to time dependent tangential stress applied at the surface. Free convection effect on the Stoke's problem for an infinite vertical plate in a dusty fluid has been discussed by Ramamurthy [4]. Mishra [2] has investigated a problem on MHD flow of an oscillating plate in absence of pressure gradient. Sharma and Sharma [6] have studied free convection non-Newtonian flow past an infinite plate with suction and constant heat flux. Unsteady free convection MHD dusty flow between heated porous vertical plates has been studied by Subeydar [10]. Recently, Singh and Rana [7] have discussed three dimensional flow and heat transfer through a porous medium. More, recently Singh et al [8] have investigated heat transfer in three dimensional MHD flow past a porous plate.

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The object of the present paper is to analyse the effect of free convective laminar flow of an electrically conducting incompressible dusty viscoelastic liquid (Walters liquid model-B) embedded with non-conducting dust particles through a porous medium past an infinite porous plate subject to a slightly sinusoidal transverse velocity distribution. Velocity field, temperature distribution, skin-friction and heat transfer have been obtained and discussed with the help of graphs.

2. FORMULATION OF THE PROBLEM. Let us consider the laminar flow of an electrically conducting, incompressible, dusty, viscoelastic (Walters liquid model-B) liquid in a porous medium past an infinite porous oscillating flat plate. The plate executes oscillations in its own plane with frequency n decreasing exponentially with time. Let x-axis is along the flow of liquid in the plane of the plate and y-axis perpendicular to it. The dust particles are assumed to be solid, spherical, non-conducting and symmetrically distributed in the flow region. A constant velocity v_0 ($v < 0$ for suction and $a > 0$ for injection) normal to the plate is applied.

A uniform magnetic field B_0 is applied perpendicular to the flow region. Since the liquid is viscoelastic, the induced magnetic field has been neglected in comparison with the applied magnetic field. It is assumed that the temperature of the plate remains steady at T_W and free stream temperature at T. Then the governing boundary layer equations for a two-dimensional unsteady, laminar, incompressible dusty viscoelastic liquid (Walters liquid model-B) can be expressed as

$$\frac{\partial u}{\partial t} - v_0 \frac{\partial u}{\partial y} = g_x \beta (T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma}{\rho} B_0^2 u - \frac{\nu}{k} u + \frac{K_1 N_0}{\rho} (v - u) - k \left(\frac{\partial^3 u}{\partial y^2 \partial t} - \frac{\partial^3 u}{\partial y^3} \right).$$

$$m \frac{\partial v}{\partial t} = K_1 (u - v) \qquad \ldots (2.2)$$

$$\frac{\partial T}{\partial t} - v_0 \frac{\partial T}{\partial y} = \frac{\lambda}{\rho c_p} \frac{\partial^2 T}{\partial y^2} \qquad \ldots (2.3)$$

The boundary conditions are :

$$u = v = v_0 (1 + \epsilon e^{-nt}), \; T = T_W (1 + \epsilon e^{-nt}) \text{ at } y = 0 \qquad \ldots (2.4)$$

$$u = v \Rightarrow v_0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty$$

where u is the velocity of the liquid, v is the velocity of the dust particles, c_p is the specific heat at constant pressure of the liquid, β is the coefficient of volume expansion, g_x is the acceleration due to gravity, ν is the kinematic viscosity of the liquid, K is the elastic

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constant, ρ is the density of the liquid, t is the time, λ is the thermal conductivity of the liquid, T is the temperature in boundary layer, B_0 is the induction of magnetic field, k is the permeability of the medium, T_∞ is the temperature of the liquid for away from the plate, σ is the electrical conductivity of the liquid, m is the mass of dust particules, N_0 is the number density of dust particles and v_0 is the suction velocity.

We introduce the following non-dimensional quantities -

$$y^* = \frac{yv_0}{\nu},\quad u^* = \frac{u}{v_0},\quad v^* = \frac{v}{v_0},\quad t^* = \frac{v_0^2}{4\nu},$$

$$K^* = \frac{Kv_0^2}{\nu^2},\quad T^* = \frac{T - T_\infty}{T_w - T_\infty},\quad \frac{\mu c_p}{\lambda} = p \text{ (Prandtl number)},$$

$$\frac{\nu^2}{v_0^2 k} = \frac{1}{k^*} \text{ (Permeability parameter)},$$

$$\frac{\sigma\nu B_0^2}{\rho v_0^2} = M \text{ (Hartmann number)},$$

$$\frac{\nu g_x (T_w - T_\infty)\beta}{v_0^3} = G \text{ (Grashoff number)}.$$

Using these non-dimensional quantities, the equations (2.1), (2.2) and (2.3) after ignoring the stars over them reduces to

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} - (M + \frac{1}{k})u + \frac{l}{w}(v - u) - \frac{1}{4}\frac{\partial u}{\partial t} - K(\frac{1}{4}\frac{\partial^3 u}{\partial y^2 \partial t} - \frac{\partial^3 u}{\partial y^3}) = -GT \quad \text{.... (2.5)}$$

$$w\frac{\partial v}{\partial t} = u - v \quad \text{... (2.6)}$$

and $$\frac{\partial^2 T}{\partial y^2} + p\frac{\partial T}{\partial y} - \frac{P}{4}\frac{\partial T}{\partial t} = 0 \quad \text{... (2.7)}$$

where $l = \frac{m N_0}{\rho}$ (mass concentration of dust particles), and $w = mv^2{}_0/4\nu K_1$.

The non-diemensional boundary conditions are

$$u = v = 1 + \epsilon e^{-nt},\quad T = 1 + \epsilon e^{-nt} \text{ at } y = 0$$
$$u = v \to 1 \quad , \quad T \to 0 \quad \text{as} \quad y \to \infty \quad \text{... (2.8)}$$

3. SOLUTION OF THE PROBLEM.

Following Lighthill [1], we assume

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$$T(y, t) = [1 - f_1(y)] + \epsilon\, e^{-nt}\, [1 - f_2(y)] \quad \ldots (3.1)$$

$$u(y, t) = g(y) + \epsilon\, e^{-nt}\, g_2(y) \quad \ldots (3.2)$$

$$v(y, t) = F_1(y) + \epsilon\, e^{-nt}\, F_2(y) \quad \ldots (3.3)$$

Substituting (3.1) - (3.3) in the equations (2.6) and (2.7), after comparing the harmonic terms we get

$$f_1'' + pf_1' = 0 \quad \ldots (3.4)$$

$$\frac{1}{P} f_2'' + f_2' + \frac{n}{4} f_2 = \frac{n}{4} \quad \ldots (3.5)$$

$$g_1 = F_1 \quad \ldots (3.6)$$

$$F_2 = \frac{g_2}{(1 - nw)}. \quad \ldots (3.7)$$

Using relations (3.6), (3.7) and (3.1) - (3.3) in the equation (2.5), we obtain

$$K g_1''' + g_1'' + g_1' - (M + \frac{1}{k})\, g_1 = (f_1 - 1)\, G \quad \ldots (3.8)$$

$$Kg_2''' + (1 + \frac{Kn}{4})\, g_2'' + g_2' - (M + \frac{1}{k} - \frac{n}{4} - \frac{ln}{1 - wn}) g_2 = (f_2 - 1)G \quad \ldots (3.9)$$

The transformed boundary conditions are :

$$f_1 = f_2 = 0,\ g_1 = a_2 = 1,\ F_1 = F_2 = 1 \text{ at } y = 0 \quad \ldots (3.10)$$

$$f_1 = f_2 \to 1,\ f = g_1 \to 1,\ g_2 = F_2 \Rightarrow 0 \text{ as } y \to \infty$$

The soluiton of equations (3.4) and (3.5) under the boundary condtions (3.10), we have

$$f_1 = 1 - \exp(-Py) \quad \ldots (3.11)$$

$$f_2 = 1 - \exp(-PH_2 y) \quad \ldots (3.12)$$

where $H_2 = \frac{1}{2}\left[1 + (1 - \frac{n}{p})^{1/2}\right]$

Following Soundalgekar [9], we assume

$$g_1 = g_{01} + Kg_{11} + 0(K^2) \quad \ldots (3.13)$$

$$g_2 = g_{02} + Kg_{12} + 0(K^2) \quad \ldots (3.14)$$

$$F_1 = F_{01} + KF_{11} + 0(K^2) \quad \ldots (3.15)$$

$$F_2 = F_{02} + KF_{12} + 0(K^2) \quad \ldots (3.16)$$

Substituting (3.13) - (3.16) and using (3.11) and (3.12) in the equations (3.6) - (3.9), we obtain

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$$f_{01} = g_{01}, \quad f_{11} = g_{11} \qquad \ldots (3.17)$$

$$F_{02} = \frac{g_{02}}{(1 - wn)} \qquad \ldots (3.18)$$

$$f_{12} = \frac{g_{12}}{(1 - wn)} \qquad \ldots (3.19)$$

$$g_{01}'' + g_{01}' - (M + \frac{1}{k}) g_{01} = - G e^{-py} \qquad \ldots (3.20)$$

$$g_{11}'' + q_{11}' - (M + \frac{1}{k}) g_{11} = - g_{01}''' \qquad \ldots (3.21)$$

$$g_{02}'' + q_{02}' - (M + \frac{1}{K} - \frac{n}{4} - \frac{ln}{(1 - wn)} g_{02} = - Ge^{-PH_2 y} \qquad \ldots (3.22)$$

$$G_{12}'' + g_{12}' - (M + \frac{1}{k} - \frac{n}{4} - \frac{ln}{(1 - 2n)}) g_{12} = - \frac{n}{4} g_{02}'' - g_{02} \qquad \ldots (3.23)$$

The corresponding boundary conditions are

$$g_{01} = 1, \quad g_{11} = 0, \quad g_{02} = 1, \quad g_{12} = 0,$$

$$F_{01} = 1, \quad F_{11} = 0, \quad f_{02} = 1, \quad F_{12} = 0, \quad \text{at} \quad y = 0 \qquad \ldots (3.24)$$

$$g_{01} \to 1, \; F_{01} \to g_{11} = g_{02} = g_{12} = F_{02} = F_{11} = F_{12} \to 0$$

as $y \to \infty$.

On solving equations (3.17) to (3.23) under the boundary conditions (3.24), after substituting in the equations (3.13) - (3.16), we have

$$g_1 = 1 + Gb_1(e^{-H_4 y} - e^{-py}) + KGb_1^2 p^3 (e^{-H_4 y} - e^{-Py}) \qquad \ldots (3.25)$$

$$g_2 = (1 + Gb_3 + KGb_3 b_5) e^{-H_6 y} - b_3 G(1 + Kb_5) e^{-PH_2 y} \qquad \ldots (3.26)$$

$$F_1 = 1 + Gb_1(e^{-H_4 y} - e^{-py}) + KGb_1^2 p^3 (e^{-H_4 y} - e^{-py}) \qquad \ldots (3.27)$$

$$F_2 = (1 + Gb_7 + KGb_9 b_3) e^{-H_6 y} - (Gb_7 + KGb_3 b_9) e^{-PH_2 y} \qquad \ldots (3.28)$$

Hence on putting the values of g_1, g_2, F_1, F_2, f_1 and f_2 in the equations (3.1), (3.2) and (3.3) we obtain the velocity of the liquid, particle and temperature of the liquid

$$u = 1 + Gb_1(1 + Kb_1 p^3)(e^{-H_4 y} - e^{-py}) + \in (1 + Gb_3 + KGb_3 b_5)$$

$$e^{-(H_6 y + nt)} - \in Gb_3(1 + Kb_5) e^{-(PH_2 y + nt)} \qquad \ldots (3.29)$$

$$v = 1 + Gb_1 (1 + Kb_1 p^3)(e^{-H_4 y} - e^{-py}) + \in (1 + Gb_7 + KGb_3 b_9)$$

$$- e^{-(H_6 y + nt)} - \in G(b_7 + Kb_3 b_9) e^{-(PH_2 y + nt)} \qquad \ldots (3.30)$$

$$T = e^{-py} + \in e^{-(PH_2 y + nt)} \qquad \ldots (3.31)$$

where,

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$$H_4 = \frac{1}{2}\left[1 + \left\{1 + 4\left(M + \frac{1}{K}\right)\right\}^{1/2}\right]$$

$$H_6 = \frac{1}{2}\left[1 + \left\{1 + 4\left(M + \frac{1}{K} - \frac{n}{4} - \frac{n}{1 - wn}\right)\right\}^{1/2}\right]$$

$$b_1 = \left[P(P-1) - \left(M + \frac{1}{K}\right)\right]^{-1}$$

$$b_3 = \left[PH_2(PH_2 - 1) - \left(M + \frac{1}{k} - \frac{n}{4} - \frac{ln}{1 - wn}\right)\right]^{-1}$$

$$b_5 = b_3 P^2 H_2^2 \left(PH_2 - \frac{n}{4}\right)$$

$$b_7 = b_3 \mid (1 - 2n)$$

$$b_9 = b_7 b_5.$$

4. SKIN-FRICTION AND HEAT TRANSFER.

Skin-friction τ_1 for the liquid is given by

$$\tau_1 = \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= b_1 G(1 + Kb_1 p^3)\,(P - H_4) - \epsilon\left[(1 + Gb_3 + Kb_3 b_5 G)H_6 - (Gb_3 + KGb_3 b_5)\,PH_2\right]e^{-nt} \quad \ldots (4.1)$$

Skin-friction τ_2 of the dust particles is given by

$$\tau_2 = \left(\frac{\partial v}{\partial y}\right)_{y=0}$$

$$\tau_2 = Gb_1\,(1 + Kb_1 p^3)\,(P - H_4) - \epsilon\left[(1 + b_7 G + KGb_3 b_9)\,H_6 - (b_7 + Kb_3 b_9)\,GPH_2\right]e^{-nt} \quad \ldots (4.2)$$

Heat transfer in terms of Nusselt number N_a is given by

$$N_a = -\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$= P(1 + \epsilon H_2)\,e^{-nt} \quad \ldots (4.3)$$

5. DISCUSSION. Velocity profiles of the liquid and dust particles for varius values of magnetic field ($M = 0.2$, 1.2 and 2.2) at $P = .71$, $G = 5.0$, $l = 0.3$, $w = 0.2$, $K = 1.0$, $\epsilon = 0.4$, $t = 1.0$, $n = 0.1$ and $k = 0.25$ are shown in figure 1. From the figure it is obvious that the velocity of the liquid and particles decreases as the intensity of magnetic field M increases. We also observe that the velocity of the

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FIG.1 VELOCITY PROFILES OF THE LIQUID AND DUST PARTICLES FOR VARIOUS VALUES OF MAGNETIC FIELD (HARTMANN NUMBER M)

(P=0·71, G=5·0, l=0·3, w=0·2, K=1·0, ϵ=0·4, t=1·0, n=0·1 and k=0·25).

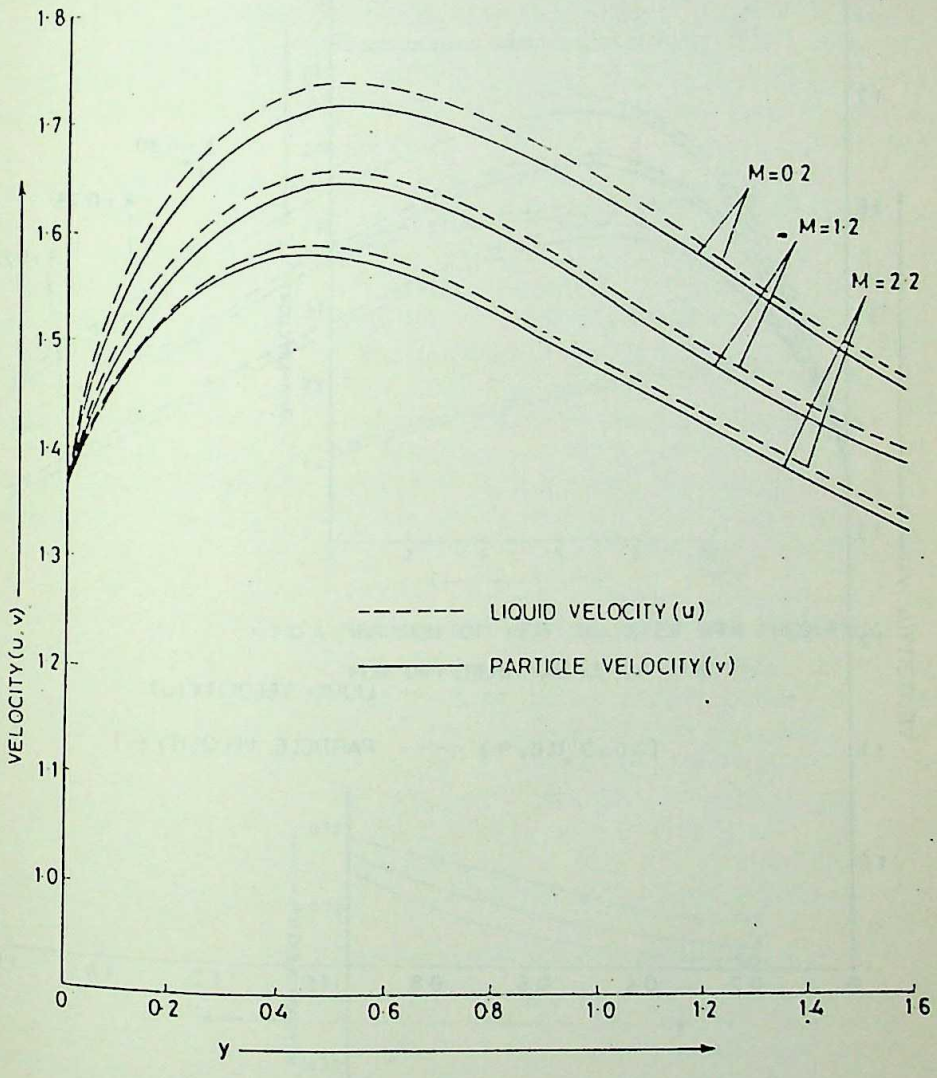

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FIG 2 VELOCITY PROFILES OF THE LIQUID AND DUST PARTICLE FOR VARIOUS VALUES OF POROSITY PARAMETER k (P=0·71, G=5·0, l=0·3, w=0·2, K=1·0, ε=0·4, t=1·0, n=0·1 and M=1·2)

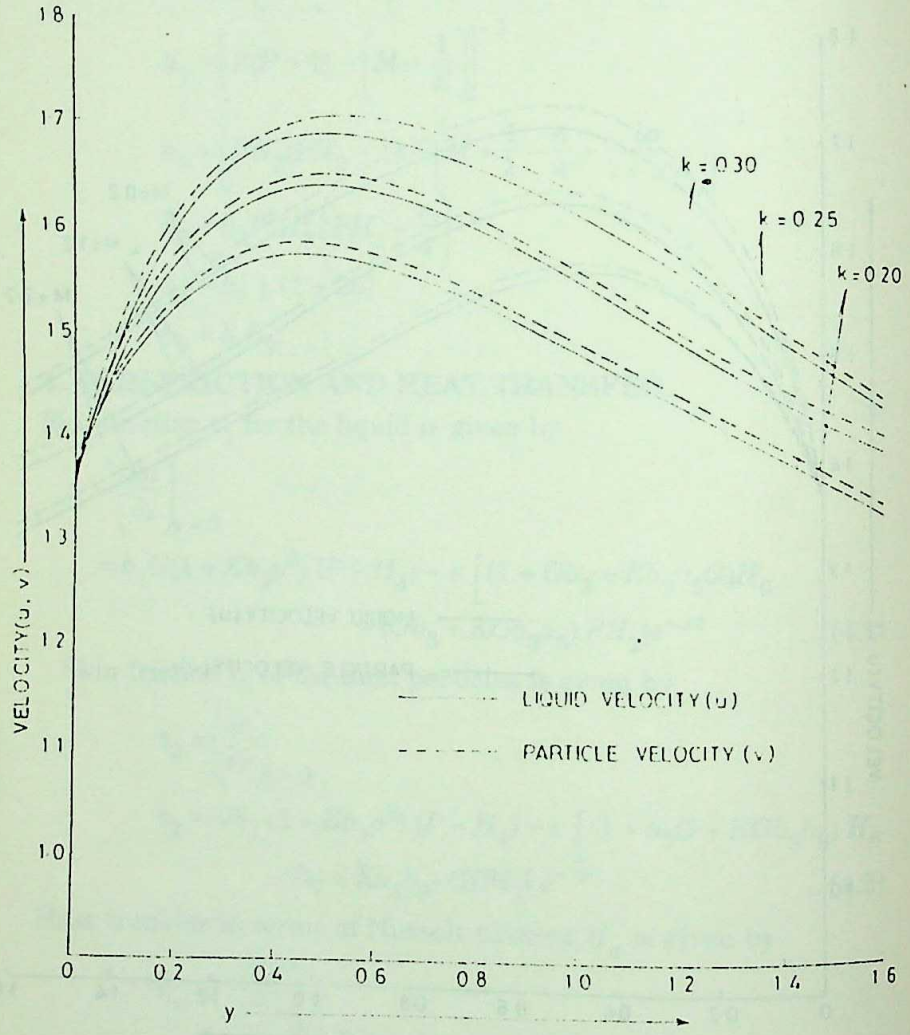

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FIG.3 VARIATION OF SKIN-FRACTION FACTOR WITH TIME FOR DIFFERENT VALUES OF GRASHOFF NUMBER G, (P=0.71, l=0.3, w=0.2, K=1.0, ϵ=0.4, n=0.1, k=0.25 and M=1.2)

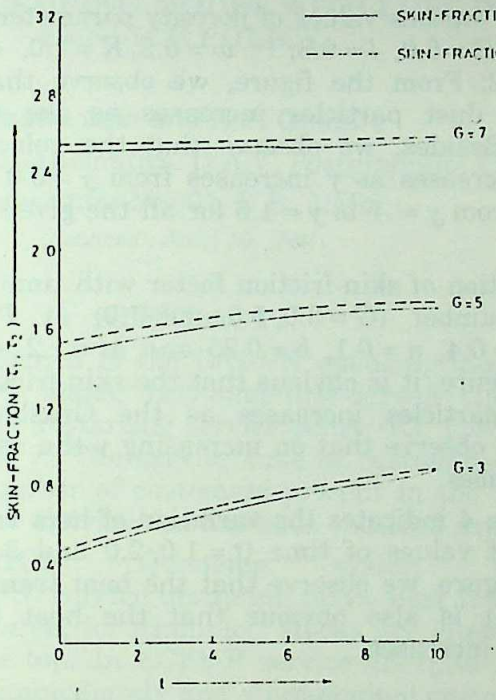
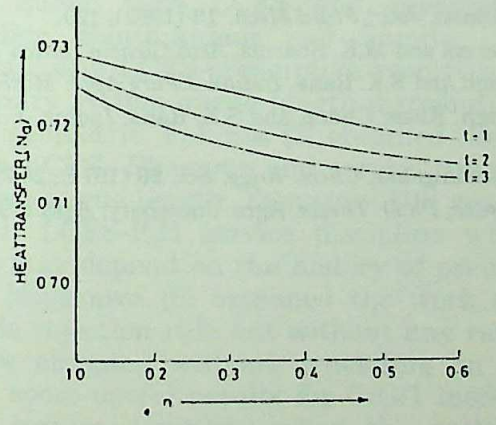


FIG.4 VARIATION OF HEAT TRANSFER WITH FREQUENCY FOR DIFFERENT VALUES OF TIME t

(P=0.71, ϵ=0.4)



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liquid and particles increases as y increases from $y = 0.0$ to $y = 0.4$ after that the velocity of the liquid particles decreases from $y = 0.4$ to $y = 1.6$ for all given magnitudes of magnetic intensity.

Figure 2 shows the velocity profiles of the liquid and dust particles for various values of porosity parameter ($k = .20$, $.25$ and $.30$) at $P = .71$, $G = 5.0$, $l = 0.3$, $w = 0.2$, $K = 1.0$, $\epsilon = 0.4$, $t = 1.0$, $n = 0.1$ and $M = 1.2$. From the figure, we observe that the valocity of the liquid and dust particles increases as the porosity parameter k increases. Besides, we observe that the velocity of the liquid and particles increases as y increases from $y = 0.0$ to $y = .4$ after that it decreases from $y = .4$ to $y = 1.6$ for all the given values of the porosity parameter.

Variation of skin-friction factor with time for different values of Grashoff number ($G = 3.0$, 5.0 and 7.0) at $P = .71$, $l = 0.3$, $w = 0.2$, $K = 1.0$, $\epsilon = 0.4$, $n = 0.1$, $k = 0.25$ and $M = 1.2$ are shown in figure 3. From the figure, it is obvious that the skin-friction factor of the liquid and dust particles increases as the Grashoff number increases. Besides, we observe that on increasing y the variation of skin-friction factor increases.

Figure 4 indicates the variation of heat transfer with frequency for different values of time ($t = 1.0$, 2.0 and 3.0) at $P = .71$, $\epsilon = 0.4$. From the figure, we observe that the heat transfer decreases as time increases. It is also obvious that the heat transfer decreases as frequency n increases.

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G/G/1/N QUEUEING MODEL WITH LCFS-P/R SERVICE POLICY

By

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ABSTRACT

This paper is devoted to the G/G/1/N queueing model in which customers are served under last-come-first-served (LCFS) queue discipline and arbitrary restarting service policy (R). The input of the is stationary with rate λ. The service time of customers depends on the queue size (i.e. number of customers present in the system). For steady-state, queue size distribution has been obtained explicitly.

1.INTRODUCTION. Queueing system with last come-first-served (LCFS) service discipline plays vital role in practice however in a limited area, for example : stocks are often refilled but also worked off at the top. In LCFS-P service discipline an arriving customer gets service immediately and a pre-empted customer restarts service when all customers arriving after him leave the system. Some authors have made efforts to tackle the queueing problems having LCFS service discipline. Kelly [5] introduced LCFS-P/R queue discipline for M/G/1 system. Yamazaki [8] analysed the G/G/1 queue with LCFS service discipline and later on, he [9] obtained invariance relations for GI/G/1 queue with LCFS and preemptive resume restarting service policy. Santhikumar and Sumita [7] discussed GI/GI/1 model 1 by imposing LCFS-preemptive restarting policy and proved that the stationary system queue length distribution jsut after a departure instant is geometric. Fakinos [2] obtained the expressions for GI/GI/1 queue with LCFS- P/resume and service time distribution depending on the system queue length. Yamazaki [10] genralized it for GI/GI/1/K system with LCFS-P/H service discipline where H is a restarting plicy which may depend on the history of premption of the restarting customers. Miyazawa [6] extended the work of Yamazaki [10] by imposing simple rejection rule but without any restriction (i.e. arriving customers are accepted without depending on queue size). Fakios [3, 4] obtained some useful results for G/G/1 model under the last-come-first served service discipline when the customers where drawn from infinite population and interarrival times and service times under stationary stochastic processes. Dijk [1] studied a finite

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LCFS buffer queueing model with batch input and non-exponential service.

In may practical applications, the customers are often drawn from finite population for example; telecommunication, computer and manufacturing systems etc. The present day demands motivate us to investigate a G/G/1/N model under LCFS/P service discipline For solution purpose the system is considered at arrival epochs and in continuous time. The steady state queue size distribution has been obtained in section 2. Section 3 deals with special case. In the last section 4, the conclusion has been drawn.

2. THE MODEL To describe our model we use the following notations :

j = Numebr of custormers waiting behind each moving customer from position $n-1$ to position n.

b = Mean value of random variable x_n,

c = Mean value of the busy period.

σ = $1 - b/c$.

N = Total number of customers present in the system.

Q_K = Queue size immediately before Kth arrival epocks.

$Q(t)$ = Queue size at time t.

$Q(n)$ = Number of customer in queue n.

r_n, p_n = Limiting probabilities distributions of queue size.

r_n, p_n = Tails probabilities of r_n and p_n respectively.

b^* = Mean actual service time at conditional probability.

$\bar{b}^*$ = Mean service time at unconditional probability.

s_n = Mean system time at conditional probability.

$\bar{s}_n$ = Mean system time unconditional probability.

w_n = Mean waiting time for type n customers.

$B_n(x)$ = Distribution function of service time.

The stochastic processes $\{Q(t) : t \geq 0\}$ and $\{Q_K : K = 1, 2, \ldots, N\}$ have limiting probability distributions :

$$p_n = \lim_{t \to \infty} p\{Q(t) = n\}$$

and $$r_n = \lim_{K \to \infty} p\{Q(n) = n\}, (n = 0, 1, 2, \ldots, N\}.$$

The corresponding tail probabilities are

$$\hat{p}_n = \sum_{i=n}^{N} p_i \qquad \ldots (1)$$

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$$r_n = \sum_{i=n}^{N} r_i \qquad \ldots (2)$$

Under LCFS/P discipline the relation between limiting probability distributions p_n and r_n are given by (Yamazaki [10])

$$p_n = \rho_n r_{n-1}, \quad n = 1, 2, \ldots, N \qquad \ldots (3)$$

where $\rho_n = \lambda b_n$ is traffic intensity.

and

$$r_n = \prod_{j=1}^{n} \sigma_j \Big/ \sum_{K=0}^{N} \prod_{j=1}^{K} \sigma_j \qquad \ldots (4)$$

We assume that in equilibrium the arrival processes for queue n is a stationary point process with rate $\lambda_n = \lambda \hat{r}_{n-1}$. The mean service time ($\bar{b}_n^*$) and mean system time ($\bar{s}_n$) are given respectively by

$$\bar{b}_n^* = \sum_{j=0}^{N-n} b_{n+j}^* \frac{r_{n-1+j}}{\hat{r}_{n-1}} \qquad \ldots (5)$$

$$\bar{s}_n = \sum_{j=0}^{N-n} s_{n+j} \frac{r_{n-1+j}}{\hat{r}_{n-1}} \qquad \ldots (6)$$

In equilibrium for non-empty queue n

$$p(Q(n) > 0) = \lambda_n \bar{b}_n^* \qquad \ldots (7)$$

so that (5) implies

$$p\{Q(1) \geq n\} = \lambda \hat{r}_{n-1} \sum_{j=0}^{N-n} b_{n+j}^* \frac{r_{n-1+j}}{\hat{r}_{n-1}}$$

Also $$\hat{p}_n = \sum_{j=0}^{N-n} \rho_{n+j}^* \; r_{n-1+j}, \qquad \ldots (8)$$

$$(n = 1, 2, \ldots, N)$$

We note that

$$p_n = \hat{p}_n - \hat{p}_{n+1}$$

$$= \rho_n^* \; r_{n-1} \qquad (n = 1, 2, \ldots, N) \qquad \ldots (9)$$

where $\rho_n^* = \lambda b_n^*$ is new traffic intensity.

Using Little's result to queue n, we get

$$E\,[Q\,(n)] = \lambda_n \bar{s}_n, \qquad \ldots (10)$$

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$$E[Q(n)] = \sum_{k=1}^{N-n} p \quad \{Q(n) \geq K\}$$

$$= \sum_{K=1}^{N-n} p\{Q(1) \geq n + K - 1\}$$

$$= \sum_{K=1}^{N-n} \hat{p}_{n+K-1} \qquad \ldots (11)$$

From equation (6) and (11), equation (10) yields

$$\sum_{j=0}^{N-n} \hat{p}_{n+j} = \lambda \sum_{j=0}^{N-n} s_{n+j} \; r_{n-i+j} \qquad \ldots (12)$$

so that

$$\hat{p}_n = s_n \, r_{n-1} \qquad \ldots (13)$$

and

$$\hat{p}_n - \hat{p}_{n+1} = \lambda \, (s_n \, r_{n-1} - s_{n+1} \, r_n) \qquad \ldots (14)$$

Now from equations (9) and (14), we get

$$r_n = \frac{(s_n - b_n^*)}{s_{n+1}} \, r_{n-1} \qquad \ldots (15)$$

Denoting

$$\sigma_n = \frac{s_n - b_n^*}{s_{n+1}} = \frac{w_n}{s_{n+1}}, \quad (n = 1, 2, \ldots, N) \qquad \ldots (16)$$

Equation (15) reduces to

$$r_n = \sigma_n \, r_{n-1} = \left(\frac{w_n}{s_{n+1}} \right) r_{n-1}, \; (n = 1, 2, \ldots, N) \ldots (17)$$

From normalizing condition

$\sum_{K=0}^{N} r_K = 1$ and equation (17), we get

$$r_0 \sum_{K=0}^{N} \prod_{j=1}^{K} \sigma_j = 1 \qquad \ldots (18)$$

Thus limiting probability distribution r_n is given by

$$r_n = \prod_{j=1}^{n} \sigma_j \Big/ \sum_{K=0}^{N} \prod_{j=1}^{K} \sigma_j, \qquad (n = 0, 1, 2, \ldots, N) \qquad \ldots (19)$$

Also equation (9) becomes

$$p_n = p_n^* \prod_{j=1}^{n-1} \sigma_j \Big/ \sum_{k=0}^{n} \prod_{j=1}^{k} \sigma_j, \quad (n = 0, 1, \ldots, N) \qquad \ldots (20)$$

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The limiting probability when there is no customer in the system is given by

$$p_0 = 1 - \sum_{n=1}^{N} \rho_n^* \prod_{j=1}^{n-1} \sigma_j \Big/ \sum_{K=0}^{N} \prod_{j=1}^{K} \sigma_j \qquad \ldots (21)$$

3. SPECIAL CASE

In the special case when service requirement is independent of queue size, $B_n(x) = B(x)$, $b_n^* = b^*$, $s_n = s$, $\sigma_n = \sigma$ and $\rho_n^* = \rho^*$, $(n = 1, 2, \ldots, N)$, we have

$$p_0 = \frac{1 - \rho^*}{1 - p^{*N+1}} \qquad \ldots (22)$$

$$r_0 = \frac{1 - \sigma}{1 - \sigma^{N+1}} \qquad \ldots (23)$$

where $= 1 - b/c$

so that

$$\hat{r}_n = (1 - b^*/c^*)\ \hat{r}_{n-1} \qquad \ldots (24)$$

and

$$r_n = r_0 (1 - b^*/c^*)^n\ b^*/c^* \qquad \ldots (25)$$

From equation (9), we get

$$P_n = \frac{(1 - \sigma)}{1 - \sigma^{N+1}} \rho^*\ \sigma^{n-1}\ (n = 1, 2, \ldots, N) \qquad \ldots (26)$$

4. CONCLUSION

The steady state queue size distribution obtained explicitly for considered G/G/1/N model is of great utilizaton due to its applications in finite buffer situations in industrial problems. The last-come-first-served queue discipline with preemption and arbitrary restarting policy are common in inventory and production systems.

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MEROMORPHIC MULTIVALENT FUNCTIONS WITH FIXED COEFFICIENTS

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ABSTRACT

In the present paper we have obtained coefficient inequality for the class $Q_k^*(p, A, B)$ which consists of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0)$$

which are regular and p-valent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Further we have shown that the class $Q_k^*(p, A, B)$ is closed under arithmetic mean and convex linear combinations. Lastly we have obtained the radius of convexity for the class $Q_k^*(p, A, B)$. Various results obtained in the present paper are shown to be sharp.

1. Introduction

Let $Q(p)$ denotes the class of functions of the type

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n} \qquad \ldots (1.1)$$

which are regular and p-valent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Further, let $Q^*(p)$ denotes the subclass of $Q(p)$ consisting of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n} \text{ and satisfying } a_{p+n} \geq 0. \qquad \ldots (1.2)$$

In [2] Uralegaddi-Ganigi introduced a class $Q^*(p, A, B)$ consisting of functions of type (1.2) and satisfying the condition

$$\left| \frac{zf'(z)/f(z) + p}{Bzf'(z)/f(z) + Ap} \right| < 1, \ |z| < 1 \qquad \ldots (1.3)$$

where

$$-1 \leq A < B \leq 1, \ A + B \geq 0.$$

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Meromorphic univalent functions have been rather extensively studied by several authors like Uralegaddi [3] and Uralegaddi- Ganigi [2]. We begin by recalling the following lemma due to Uralegaddi-Ganigi [2].

Lemma 1 Let the function $f(z)$ be defined by (1.2). Then the function $f(z)$ is in the class $Q_k^*(p, A, B)$ if and only if

$$\sum_{n=0}^{\infty} \{(1+B)n + (2+A+B)p\}\, a_{p+n} \le p(B-A). \quad \ldots (1.4)$$

In view of Lemma 1, we observe that $f(z)$ given by (1.2) in the class $Q_k^*(p, A, B)$ satisfy

$$a_p \le \frac{(B-A)}{(2+A+B)}. \quad \ldots (1.5)$$

Hence we may take

$$a_p = \frac{(B-A)k}{(2+A+B)}, \; 0 \le k \le 1. \quad \ldots (1.6)$$

Let $Q_k^*(p, A, B)$ denotes the subclass of $Q_k^*(p, A, B)$ consisting of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} \quad \ldots . (1.7)$$

where

$$a_{p+n} \ge 0 \text{ and } 0 \le k \le 1.$$

In the present paper we have obtained coefficient inequality for the class $Q_k^*(p, A, B)$. Further we show that the class $Q_k^*(p, A, B)$ is closed under arithmetic mean and convex linear combinations. Lastly we have obtained radius of convexity for the class $Q_k^*(p, A, B)$. Various results obtained in this paper are shown to be sharp. Techniques used are similar to those of Silverman and Silvia [1].

Main Results

2. Coefficient Inequality

Theorem 1 Let the function $f(z)$ be defined by (1.7). Then is in the class $Q_k^*(p, A, B)$ if and only if

$$\sum_{n=1}^{\infty} \{(1+B)n + (2+A+B)p\} a_{p+n} \le p(B-A)(1-k). \quad \ldots (2.1)$$

The result is sharp.

Proof Putting

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$$a_p = \frac{(B-A)\,k}{2+A+B}, \quad 0 < k \le 1 \qquad \ldots (2.2)$$

in (1.4) and simplifying we get result. The result is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \frac{(B-A)\,p(1-k)}{\{(1+B)n + (2+A+B)p\}} z^{p+n}, \quad n \ge 1. \qquad \ldots (2.3)$$

Corollary Let the function $f(z)$ given by (1.7) be in the class $Q_k^*(p, A, B)$. Then

$$a_{p+n} \le \frac{(B-A)\,p(1-k)}{\{(1+B)n + (2+A+B)p\}}, \; n \ge 1. \qquad \ldots (2.4)$$

3. Closure Theorem

In this section we shall show that the class $Q_k^*(p, A, B)$ is closed under arithmetic mean and convex linear combinations.

Theorem 2 Let

$$f_j(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \sum_{n=1}^{\infty} a_{p+n,j} z^{p+n} (a_{p+n,j} \ge 0) \qquad \ldots (2.5)$$

be in the class $Q_k^*(p, A, B)$ for every $j = 1, 2, \ldots, m$, then the function

$$g(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \; (b_{p+n} \ge 0) \qquad \ldots (2.6)$$

is also a member of $Q_k^*(p, A, B)$, where

$$b_{p+n} = \frac{1}{m} \sum_{j=1}^{m} a_{p+n,j}.$$

Proof Since $f_j(z) \in Q_k^*(p, A, B)$ it follows from Theorem 1 that

$$\sum_{n=1}^{\infty} \{(1+B)n + (2+A+B)p\}\, a_{p+n,j} \le (b-A)p\,(1-k) \qquad \ldots (2.7)$$

for every $j = 0, 1, 2, \ldots, m$. Hence

$$\sum_{n=1}^{\infty} \{1+B)n + (2+A+B)p\}\, b_{p+n}$$

$$= \sum_{n=1}^{\infty} \{(1+B)n + (2+A+B)p\} \left[1/m \sum_{j=0}^{m} a_{p+n,j} \right]$$

$$= \frac{1}{m} \sum_{j=0}^{m} \sum_{n=1}^{\infty} \{(1+B)n + (2+A+B)p\}\, a_{p+n,j}$$

$$\le (B-A)p\,(1-k)$$

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$$\leq (B-A)\, p(1-k)$$

and the result follows.

Theorem 3 Let

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)}\, z^p$$

and

$$f_{p+n}(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \frac{(B-A)\, p(1-k)}{\{(2+A+B)p + (1+B)n\}}\, z^{p+n},\ (n \geq 1).$$

Then $f(z) \in Q^*(p, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z).$$

where

$$\lambda_{p+n} \geq 0 \text{ and } \sum_{n=-1}^{\infty} \lambda_{p+n} = 1.$$

Proof Let

$$f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z),$$

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)}\, z^p + \sum_{n=1}^{\infty} \frac{(B-A)\, p(1-k)}{\{(1+B)n + (2+A+B)p\}}\, z^{p+n}.$$

Since

$$\sum_{n=1}^{\infty} \frac{(B-A)\, p(1-k)\ \lambda_{p+n}}{\{(1+B)n + (2+A+B)p\}} \cdot \frac{\{(1+B)n + (2+A+B)p\}}{(B-A)\, p(1-k)}$$

$$= \sum_{n=1}^{\infty} \lambda_{p+n} = 1 - \lambda_p \leq 1.$$

Hence, by Theorem 1, $f(z) \in Q_k^*(p, A, B)$.

Conversely, suppose that

$$f(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n}\ (a_{p+n} \geq 0)$$

is in the class $Q_k^*(p, A, B)$. Then by using (2.4) we get

$$a_{p+n} \leq \frac{(B-A)\, p(1-k)}{\{(1+B)n + (2+A+B)p\}},\ n \geq 1.$$

Setting

$$\lambda_{p+n} = \frac{\{(1+B)n + (2+A+B)p\}}{(B-A)\, p(1-k)}\, a_{p+n}\ (n \geq 1).$$

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and

$$\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_{p+n}$$

we have

$$f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z).$$

4. Radius of Convexity

Theorem 4

Let $f(z)$ defined by (1.7) be in class $Q^*k(p, A, B)$. Then $f(z)$ is mermorphically p-valent convex in $0 < |z| < r = r(p, A, B, K)$, is the largest value for which

$$\frac{3p^2 (B-A)\, kr^{2P}}{(2+A+B)} + \frac{(p+n)(3p+n)(B-A)\, p(1-k)}{\{(1+B)n + (2+A+B)p\}} r^{2p+n} \le p^2,$$

$(n = 1, 2, \ldots).$

The result is sharp for the function

$$f_{p+n}(z) = \frac{1}{z^p} + \frac{(B-A)k}{(2+A+B)} z^p + \frac{(B-A)\, p(1-k)}{\{(1+B)n + (2+A+B)p\}} z^{p+n},$$

for some n.

Proof It is sufficient to show that

$$\left| \frac{\{zf'(z)\}' + pf'(z)}{f'(z)} \right| \le p \quad \text{for} \quad 0 < |z| \; r(p, A, B, k).$$

Note that

$$\left| \frac{[zf'(z)]' + pf'(z)}{f'(z)} \right|$$

$$\le \frac{\dfrac{2p^2(B-A)k}{(2+A+B)} r^{2p} + \sum_{n=1}^{\infty} (p+n)(2p+n)\, a_{p+n}\, r^{2p+n}}{p - \dfrac{(B-A)kp}{(2+A+B)} r^{2p} - \sum_{n=1}^{\infty} (p+n)\, a_{p+n}\, r^{2p+n}} \le p$$

for $0 < |z| \le r$ if

$$3p^2 \frac{(B-A)k}{2+A+B)} r^{2p} + \sum_{n=1}^{\infty} (p+n)(3p+n)\, a_{p+n}\, r^{2p+n} \le p^2.$$

Since $f(z) \in Q^*k(p, A, B)$, we may take

$$a_{p+n} = \frac{(B-A)\, p(1-k)\, \lambda_{p+n}}{\{(1+B)n + (2+A+B)p\}}, \quad \sum_{n=1}^{\infty} \lambda_{p+n} \le 1.$$

For each fixed r, choose an integer $n = n(r)$ for which

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$\frac{(p+n)(3p+n)}{(2+A+B)} r^{2p+n}$ is maximal. Then,

$$\sum_{n=1}^{\infty} (p+n)(3p+n)\, a_{p+n}\, r^{2p+n} \leq \frac{(p+n)(3p+n)(B-A)(1-k)\, r^{2p+n}}{\{(1+B)n+(2+A+B)p\}}.$$

Now find the value $r_0 = r_0(p, A, B, k)$ and corresponding $n(r_0)$ so that

$$\frac{3p^2(B-A)k}{(2+A+B)} r_0^{2p} + \frac{(p+n)(3p+n)(B-A)(p(1-k)}{\{(1+B)n+(2+A+B)p\}} r_0^{2p+n} = p^2.$$

This is the value of r_0 for which $f(z)$ is convex in $0 < |z| < r_0 = r_0(p, A, B, k)$.

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ELECTROGRAVITATIONAL INSTABILITY OF A FLUID CYLINDER UNDER MODULATING VARYING ELECTRIC FIELD ON UTILIZING THE ENERGY PRINCIPLE

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ABSTRACT

The electrogravitational instability of a fluid circular jet dispersed in a self-gravitating tenuous medium of negligible motion pervaded by modulating general varying electric field has been developed on utilizing the Lagrangian energy principle. The fundamental equations describing the problem are deriving and solved in the unperturbed and perturbed states, the total changes in the electric, kinetic and self-gravitating energies are computed. The Lagrangian energy principle technique has been used and a result it is found that the system is governing by a second order integro-differential Mathieu euqation. Several categories have been analysed via this equation. The gravitational force is only destabilizing for small axisymmetric perturbation. The internal electric field penetrated the fluid cylnder has no direct influence on the stability of the fluid while the exterior longitudinal and transverse electric fields are being stabilizing or destabilizing according to restrictions. Resonance domains are appeared due to the field periodicity and in some regions the stability conditions depend only on the field frequency. The electric field frequency is stabilizing in a small axi-symmetric region and destabilizing otherwise. The amplitude of the modulating electric field could be fully stabilizing, under certain restrictions, and suppressing the destabilizing character of the other physical parameters and hence stability arises.

Numeruous reported works could be recovered with appropriate choices as limiting cases.

1. Introduction

The in stability of cylindrical fluid column endowed with surface tension or/and acted upon external forces such as electrodynamic or electromagnetic forces has been involved in several texts by a lot of researchers. Referring to these pioneering works see.

{[2],[3],[4],[6],[9],[13],[14],[15],[16],[21],}

The response of the axisymmetric instability of a self-gravitating cylinder ambient with self-gravitating vacuum was due to Chandrasekhar and Fermi [51] It has a correlation with

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understanding the dynamical behaviour of the spiral arms of galaxies and sun spots. Chandrashekhar [4] studied such a problem in details for different cases in several literature. He [4] summarised his results along with those of others for different problems of different configuration models. Radwan [17] has extended that work [5] by studying the stability of a self- gravitating fluid cylinder dispersed in a self -gravitating fluid of different density. In the works [5] and [17] it is used totally different techniques where in the latter we have used the principle of energy.

The influence of the electrodynamic force on the self-gravitating fluid cylinder has been examined for first time by Radwan [18] . In this recent work [18] both the self-gravitating fluid and the surrounding vacuum are assumed to be pervaded by uniform (constant) electric fields. The aim of the present work is to investigate the electro-dynamic stability of a self-gravitating fluid cylinder such that the electric fields not only varying but also modulating with the same periodicity. This will be done for all axisymmetric and non-axisymmetric modes of perturbation, on using the Lagrangian energy principle technique.

2. Formulation of the Problem

Consider a dielectric self-gravitating fluid cylinder (of radius R_0) ambient with a tenuous dielectric medium of negligible motion. ε^i is the dielectric constant of the fluid matter and idem ε^e for the surrounding medium where from now on the superscripts i and e indicate interior and exterior the fluid jet. The fluid is assumed to be non-viscous, incompressible and of uniform mass density ρ. The fluid cylinder is being pervaded by the modulating electric field

$$\boldsymbol{E}_0^i = (0, 0, 1)E_0 \cos(w\,t) \qquad (1)$$

while the surrounding medium is assumed to be penetrated by the modulating general varying electric field.

$$\boldsymbol{E}_0^e = (0, \beta R_0\, r^{-1}, \alpha)E_0 \cos(w\,t) \qquad \ldots (2)$$

where E_0 and w is the intensity and frequency of the electric field and α, β are parameters satisfying certain conditions. The components of $\boldsymbol{E}_0^i$ and $\boldsymbol{E}_0^e$ are considered along the utilizing cylindrical polar coordinates (r, ϕ, z) with the z-axis coinciding with the axis of the fluid cylinder. The fluid is acting up on the intertia, electrodynamic, pressure gradient and self- gravitating forces while the surrounding medium of the fluid is acting up on the self-gravitating and electrodynamic forces. We assume,initially, that there are no surface charges at the boundary surface and therefore the surface charge density will be zero during the perturbation. We also assume that the quasi- static approximation is valid for the problem under consideration.

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The fundamental basic equation required for describing and analysing such kind of problems are coming out from the combination of the ordinary hydrodynamic equations together with those of Newtonian gravitational theory and with Maxwell's electrodynamics equations. For the problem under consideration, these basic equations may be formulated as follows.

In the fluid region :

The gravitational electrofluid dynamic vector equation of motion

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{u}\right) = (\varepsilon^i/2)\,\nabla(\boldsymbol{E}^i \,.\, \boldsymbol{E}^i) - \nabla_{\boldsymbol{p}} - \rho\nabla\Phi^i \qquad \ldots (3)$$

The equation of continuity expressing the conversation of mass

$$\nabla \,.\, \boldsymbol{u} = 0. \qquad \ldots (4)$$

The self-gravitating Poisson's equation

$$\nabla^2\Phi^i = 4\pi G\rho. \qquad \ldots (5)$$

The Maxwell's electrodynamic equations

$$\nabla \,.\, (\varepsilon^i\, \boldsymbol{E}^i) = 0, \quad \nabla \times \boldsymbol{E}^i = 0 \qquad \ldots (6), (7)$$

In the surrounding tenuous medium :

The Laplace's equation for the gravitational potential

$$\nabla^2\, \Phi^\varepsilon = 0. \qquad \ldots (8)$$

Maxwell's electrodynamic equations

$$\nabla \,.\, (\varepsilon^e\, \boldsymbol{E}^e) = 0, \;\; \nabla \times \boldsymbol{E}^e = 0. \qquad \ldots (9), (10)$$

Here p and $\boldsymbol{u}$ are the fluid kinetic pressure and velocity vector, G is the gravitational constant, Φ^i is the gravitational potential in the fluid region and idem Φ^ε in the surrounding medium and $\boldsymbol{E}$ is the electric field intensity.

3. Linearization and Solutions

In order to analyse such kind of study we assume that the fluid boundary surface is acting up on a sinusoidal deformation and consequently the location of the perturbed interface could be given in the form.

$$r = R_0 + \gamma(t)\cos(kz) + m\phi) \qquad \ldots (11a)$$

Here $\gamma(t)$ is, some function of time t, the surface diaplacement, k (any real number) is the longitudinal wavenumber and m (an integer) is the azimuthal wavenumber. The second term in the right side of (11) is the elevation of the surface wave measured from the unperturbed position. Based on the perturbation form (11 a), for small departure from the equilibrium state every physical variable quantity $Q(r, \phi, z)$ could be expressed as

$$Q(r, c, z; t) = Q_0(r) + \gamma(t)\, Q_1(r, c, z) \qquad \ldots (11b)$$

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where Q_1 is the change in Q due to a perturbation.

It is intended to investigate the stability of this problem using an analytical perturbation technique on the basis of the Lagrangian energy principle. It is noted that the Lagrangian function L is constructed as

$$L = \Omega_1 - W_1 - V_1 \qquad \ldots (12)$$

and the Lagrangian second order differential equation is being

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\gamma}}\right) - \frac{\partial L}{\partial \gamma} = 0$$

where $\dot{\gamma}$ is the Lagrangian variable for the present problem and where the dot over γ means that γ is being differentiated with respect to time. The physical quantities Ω_1, W_1 and V_1 are the changes in Ω, W and V due to the perturbation of the boundary surface (11 a) with Ω is the total kinetic energy, W is the total electrical energy and V is the gravitational potential energy. One have to refer here that the quantities with subscript 1 mean quantities in the perturbation state while those with subscript o mean their value in equilibrium state.

In the perturbation state, the basic electrodynamic equations (6), (7), (9) and (10) degenerate to

$$\nabla \times \boldsymbol{E}_1^{i,e} = 0, \quad \nabla \cdot (\varepsilon E_1^{i,e}) = 0 \qquad \ldots (14a, b)\ (15a, b)$$

From the viewpoint of the vector analysis, equations (14,a,b) mean that $\boldsymbol{E}_1^{i,e}$ could be derived from scalar electrical potentials.

$$\boldsymbol{E}_1^{i,e} = -\nabla \psi_1^{i,e} \qquad \ldots (16a, b)$$

Combining these vector equations together with equations (15 a, b), the electrical potentials $\psi_1^{i,e}$ satisfy the Laplace's equations.

$$\nabla^2 \psi_1^{i,e} = 0 \qquad \ldots (17a, b)$$

In cylindrical polar coordinate (r, ϕ, z) these equations may be rewritten as

$$r^{-1}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_1^{i,e}}{\partial r}\right) + r^{-1}\frac{\partial}{\partial \phi}\left(r^{-1}\frac{\partial \psi_1^{i,e}}{\partial \phi}\right) + r^{-1}\frac{\partial}{\partial z}\left(r\frac{\partial \psi_1^{i,e}}{\partial z}\right) = 0 \qquad \ldots (18a, b)$$

By revorting to the linear perturbation technique and based on the space time $(r, \phi, z; t)$ -dependence, the perturbed quantities ψ_1^i, ψ_1^e, ψ_1^i ψ_1^e etc. can be expressed as $\cos(kz + m\phi)$ time an amplitude functon of r viz.,

$$\psi_1^{i,e}(r, c, z; t) = \gamma(t)\, \psi_1^{i,e}(r)\ \cos(kz + m\phi) \qquad \ldots (19a, b)$$

Inserting (19 a, b) into (18 a, b) we obtain

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$$r^{-1}\frac{d}{dr}\left(r\frac{d\psi_1^{i,e}}{dr}\right)-(k^2+m^2r^{-2})\,\psi_1^{i,e}=0 \qquad \ldots (20a, b)$$

The solutions of the second order ordinary differential equations (20 a,b) are given in terms of Bessel's functions with imaginary argument. Under the present circumstances, the non-singular for ψ_1^i as $r \to 0$ interior the fluid cylinder and for ψ_1^e as $r \to \infty$ exterior it are given by

$$\psi_1^i(r) = A^i(t)\, I_m(kr) \qquad \ldots (21)$$

$$\psi_1^e(r) = A^e(t)\, K_m(kr). \qquad \ldots (22)$$

where A^i and A^e are time dependent functions of integrations to be determined; $I_m(kr)$ and $K_m(kr)$ are the modified Bessel's functions of first and second kind of order m.

The basic gravitational potential equations (5) and (8) reduce to the following equations in the unperturbed state

$$\nabla^2 \Phi_0^i = 4\pi G\rho \qquad \ldots (23)$$

$$\nabla^2 \Phi_0^e = 0. \qquad \ldots (24)$$

These equations are solved on using the unperturbed simplifications of longitudinal and azimuthal symmetries $\partial/\partial z = 0$ and $\partial/\partial c = 0$. The solutions are matched across the boundry surface at $r = R_0$. Apart from the singular solutions the solution for Φ_0^i as $r \to 0$ interior the fluid cylinder and that one for Φ_0^e exterior the cylinder as $r \to \infty$ are given by

$$\Phi_0^i = \pi G\rho r^2 \qquad \ldots (25)$$

$$\Phi_0^e = \pi\rho\, GR_0^2 + 2\pi G\rho\, R_0^2 \log(r/R_0)^2 + C_0 \qquad \ldots (26)$$

where C_0 is an additive constant with which we need not be further concerned.In the perturbed state, the basic equations (5) and (8) degenerate to the Laplace's equations

$$\nabla^2 \Phi_1^{i,e} = 0. \qquad \ldots (27a, b)$$

These equations could be solved on using similar steps as those which have already been used for solving equations (17 a, b). The non-singular solutions for Φ_1^i and Φ_1^e are given by

$$\Phi_1^i(r, \phi, z; t) = B^i\, \gamma(t)\; I_m(kr) \cos(kz + m\phi) \qquad \ldots (28)$$

$$\Phi_1^e(r, \phi, z; t) = B^e\, \gamma(t)\, K_m(kr) \cos(kz + m\phi) \qquad \ldots (29)$$

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where B^i and B^e are constants of integration to be identified. It is worthwhile to mention here that we have originally considered departures from an unperturbed right cylindrical shape of an incompressible fluids. For this reason the argument of the sinusoidal acting wave $\sin(kz + m\phi + n\pi/2)$ where n is an integer appeared as $\cos(kz + m\phi)$ in the solutions (21),(22),(28)and (29).

Now, since the Lagrangian coordinate γ is a function of time t, each element of the fluid will execute a motion. Such a motion may be derived from the Lagrangian displacement

$$\boldsymbol{u} = \partial\xi/\partial t. \qquad \ldots (30)$$

However, taking the divergence of the perturbed equation of motion which resulting equation (3), linearizing and using the incompressibility condition ($\nabla \cdot \boldsymbol{u}_1 = 0$), we obtain

$$\nabla^2 S_1 = 0, \quad S_1 = p_1/\rho + \Phi^i - (\varepsilon^{i/2})\,(\boldsymbol{E}\boldsymbol{E})^i_1. \qquad \ldots (31, 32)$$

The non-singular solution of (31) is given by

$$S_1(r, \phi, z; t) = C^i\, \gamma(t)\, I_m(kr) \cos(kz + m\phi) \qquad \ldots (33)$$

where C^i is an unspecified constant of integration.

4. Boundary Conditions

The solutions (19 a,b), (21),(22),(25) to (29) and (33) of the fundamental equations (3) to (10) must satisfy certain boundary conditions across the perturbed interface (11 a) at $r = R_0$. These appropriate boundary conditions could be stated as follows.

4.1. The electrodynamic conditions

(1) The electric potential ψ must be continuous across the perturbed interface (11 a) at $r = R_0$. This condition, on using (21) and (22), yields

$$A^e(t) = [I_m(x)/K_m(x)]\, A^i(t) \qquad \ldots (34)$$

where $x(=kR_0)$ is the dimensionless longitudinal wavenumber.

(2) The nomral component of the electric displacement must be also continuous across the perturbed boundary surface (11 a) at $r = R_0$ i.e.

$$\boldsymbol{N} \cdot (\varepsilon^i \boldsymbol{E}^i - \varepsilon^e \boldsymbol{E}^e) = 0. \qquad \ldots (35)$$

Here $\boldsymbol{N}$ is, the outward unit vector normal to the perturbed interface (11 a), given by

$$\phi = \nabla F(r, \phi, z; t)/|\nabla F(r, c, z; t)| \qquad \ldots (36)$$

where $F(r, c, z; t) = 0$ is the equation of the boundary surface given by

$$F(r, \phi, z; t) = r - R_0 - \gamma(t) \cos(kz + m\phi) \qquad \ldots (37)$$

Thus

$$\boldsymbol{N} = \boldsymbol{N}_0 + \boldsymbol{N}_1$$

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$$= (1, 0, 0) + (0, -mR_0, k)\ \gamma(t) \cos(kz + m\phi) \quad \ldots (38)$$

By the use of (1),(2),(11 a), (16 a,b), (19 a,b), (21), (22) and (38) for the condition (35), we obtain

$$\varepsilon^e K_m'(x) A^3 - \varepsilon^i I_m'(x) A^i = \left(\frac{ix}{k}(\varepsilon^i - \varepsilon^e) - \frac{im\beta}{k}\varepsilon^e\right) E_0 \cos wt \quad \ldots (39)$$

Solving (34) and (39) for $A^i(t)$ and $A^e(t)$ we get

$$A^i(t) = \frac{[iX(\varepsilon^i = \varepsilon^e) - i\beta\varepsilon^e]K_m(x)}{k[\varepsilon^e I_m(x)K_m'(x) - \varepsilon^i I_m'(x)K_m(x)]} E_0 \cos(w\,t) \quad \ldots (40)$$

$$A^e(t) = \frac{ix(\varepsilon^i - \varepsilon^e) - i\beta\varepsilon^e]\ I_m(x)}{k[\varepsilon^e I_m(x)K_m'(x) - \varepsilon^i I_m'(x)K_m(x)]} E_0 \cos(w\,t). \quad \ldots (41)$$

4.2. Self-gravitating Conditons

(*i*) The gravitational potential Φ must be continuous across the perturbed fluid interface (11 a) at $r = R_0$. This condition gives

$$B^i I_m(x) = B^e K_m(x). \quad \ldots (42)$$

(*ii*) The derivative of the self-gravitating potential Φ must also be continuous across the surface (11 a) $r = R_0$. This condition, on using (11 a), (25),(26), (28) and (29) yields

$$B^i I_m'(x) - B^e K_m'(x) = 4\pi\rho\, G/k. \quad \ldots (43)$$

Solving (42) and (43) for B^i and B^ewe get

$$B^i = 4\pi\rho G R_0 K_m(x), \quad B^e = 4\pi\rho G R_0 I_m(x) \quad \ldots (44)$$

where use have been made of the Wronskian

$$W_m(I_m(x), K_m(x)) = I_m(x)\, K_m'(x) - I_m'(x)\, K_m(x) = -x^{-1} \quad \ldots (46)$$

in obtaining (44) and (45).

4.3. The Kinematic Condition

The normal component of the velocity of the fluid particles must be compatible with the velocity of the perturbed surface (11 a) at $r = R_0$ i.e.

$$u_{1r} = \frac{\partial r}{\partial t}. \quad \ldots (47)$$

By the use of the linearized form of the equation of motion (3)

$$\rho \frac{\partial \boldsymbol{u}_1}{\partial t} = -\nabla S_1 \quad \ldots (48)$$

and equations (30) to (33) we obtain

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$$\xi_1 = -(C^i/\rho)\,\gamma(t)\,kI_m'(kr)\cos(kz + m\phi). \qquad \ldots (49)$$

By an appeal to (11 a) and (47) to (49), the coefficient C^i is completely determined and the perturbed velocity vector of the fluid is being

$$\boldsymbol{u}_1 = [R_0^2/(x\,I_m'(x))]\,\frac{\partial\gamma(t)}{dt}\,\nabla\,[I_m(kr)\cos(kz + m\phi)] \qquad \ldots (50)$$

5. Computation of Kinetic and Poetntial Energies

The change in the total kinetic energy Ω_1 (per unit length) of the fluid jet associated with the motion specified by (30) is given by

$$\Omega_1 = \tfrac{1}{2}\rho \int_0^{R_0}\int_{kz=0}^{2\pi}\int_Q^{2\pi} (\boldsymbol{u}.\boldsymbol{u})_1\, r\,dr\,\frac{dkz}{2\pi}\,dc \qquad \ldots (51)$$

$$= (\pi\rho R_0^2/(2k^2))\,[I_m'(x)]^{-2}\left(\frac{d\gamma(t)}{dt}\right)^2 J_m(y) \qquad \ldots (52)$$

where the integral $J_m(y)$ is defined by

$$J_m(y) = \int_0^x [(I_m'(y))^2 + (1 + m^2y^{-2})\,(I_m(y))^2]\,y\,dy \qquad \ldots (53)$$

This integration has been carried out on using the identity (which follows from Bessel's equations)

$$\frac{d}{dy}[y\,f_m(y)\,f_m'(y)] = y[f_m'(y))^2 + (1 + m^2\,y^{-2})\,(f_m(y))^2] \qquad \ldots (54)$$

where $f_m(y)$ stands for both the functions $I_m(x)$ and $K_m(x)$, and therefore

$$J_m(y)\,|_{r=R_0} \equiv x\,I_m(x)\,I_m'(x) \qquad \ldots (55)$$

Consequently

$$\Omega_1 = \pi\rho\,R_0^4\left(\frac{I_m(x)}{2x\,I_m'(x)}\right)\left(\frac{d\gamma(t)}{dt}\right)^2. \qquad \ldots (56)$$

Now suppose that the amplitude γ of the surface deformation (11 a) is increased by the increment $\delta\gamma$, then due to this infinitisimal increment in the amplitude of deformation the change δV in the gravitational potential energy can be determined by evaluating the work done during the displacement of the matter required to produce the change in γ. To evaluate this work it is necessary to specify quantitavely the redistribution which does take place. Arbitrary deformation of an incompressible fluid can be thought of as resulting from the Lagrangian displacement ζ_1 applied to each point of the fluid medium. We assume that the perturbed motion is irrotational and this is in fact because we have considered in the unperturbed state that

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the fluids are incompressible and non-viscous. Therefore, the Lagrangian displacement of the fluid could be expressed as

$$\zeta_1 = \nabla G_1. \qquad \ldots (57)$$

Combining equation (57) together with the incompressible condition, we find that the displacement poetential G_1 satisfies Laplace's equation

$$\nabla^2 G_1 = 0. \qquad \ldots (58)$$

The solution of this equation on using steps similar to those used for solving equations (17 a,b) is given by

$$G_1 = C_1\, \gamma(t)\, I_m(kr) \cos(kz + m\phi). \qquad \ldots (59)$$

The constant of integration C_1 could be determined by applying the condition states that the radial component of $\zeta_1$must reduce to $\gamma(t) \cos(kz + m\phi)$ at $r = R_0$. Therefore

$$\zeta_1 = \frac{R_0^2}{x\, I_m'(x)}\, \gamma(t)\, \nabla[I_m(kr) \cos(kz + mc)]. \qquad \ldots (60)$$

Hence the corresponding displacement $\delta\zeta_1$ which must be applied to each point of the fluid in order to increase the amplitude of the deformation by $\delta\gamma$ is given by

$$\delta\zeta_1 = \frac{R_0^2\, \delta\gamma(t)}{xI_m'(x)}\, \nabla\, [I_m(kr) \cos(kz + m\phi)]. \qquad \ldots (61)$$

Now, due to that additional deformation $\delta\gamma$, the change in the total self-gravitating potential energy δV_1 (per unit length) can be identified by integrating the work done by the displacements $\delta\zeta_1$ in the gravitational potential Φ_1^i. This means that

$$\delta V_1 = 2\pi\rho << \int_0^{R_0 + \gamma(t) \cos(kz + mc)} (\delta\zeta_1 \,.\, \nabla\, \Phi_1^i)\, r\, dr >> \qquad \ldots (62)$$

where the angular brackets mean that the quantity enclosed should be averaged over c and z. Substituting from (28), (44) and (61) into (62) and carrying out the required integration we get

$$\delta V_1 = 2\, \gamma(\delta\gamma)\, \rho G R_0^4\, \pi^2\, [\rho - \frac{2\rho K_m(x)}{x\, I_m'(x)}\, J_m(y)] \qquad \ldots (63)$$

where $J_m(y)$ is given by (53) or rather by (55), Inserting (55) into (63) and integrating the resulting expression from zero to γ we get

$$V_1 = -\,2\, \pi^2\, G\, \rho^2\, R_0^4\, (I_m(x)\, K_m(x) - \tfrac{1}{2})\, \gamma^2. \qquad \ldots (64)$$

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which gives the required change in the total self-gravitating potential energy due to the deformation (11 a).

Now,our duty is to determine the change in the total electrical energy W_1 of the dielectric gravitational fluid cylinder dispersed in a dielectric tenuous medium of negligible motion.

We consider an electrostatic field $\boldsymbol{E}_0^e$ which has been established in the dielectic tenuous medium of dielectric constant ε^e. We assume that a dielectric body (a fluid cylinder) of a dielectric constant ε^i has been submerged into the field while the sources of $\boldsymbol{E}_0^e$ are maintained constant. Hence the electric energy of the dielectric body (fluid cylinder) in the electric field is given (by Stratton [20] as

$$W_0 = \frac{1}{2}(\varepsilon^e - \varepsilon^i) \int (\boldsymbol{E}_0^e . \boldsymbol{e}_0^i)\, r\, dr \qquad \ldots (65)$$

where $\boldsymbol{E}_0^i$ is the modified field distribution when the dielectric fluid column has been submerged in the medium. If the fluid cylinder is deformed and the electric field distribution now becomes, $\boldsymbol{E}_1^i$, then the electric energy of the deformed fluid cylinder is given by

$$W = \frac{1}{2}(\varepsilon^e - \varepsilon^i) \iiint (\boldsymbol{E}_0^e . \boldsymbol{E}_1^i)\, r\, dr\, d\phi\, dkz. \qquad \ldots (66)$$

Therefore, the change in the electric energy due to the performed deformation (11 a) is being

$$W_1 = W - W_0. \qquad \ldots (67)$$

Substituting from (1),(2),(16 a), (19 a), (21) and (40) into (65) and (66) and carrying out the required integrations we obtain

$$W_0 = (E_0^2\, \alpha^2/2)(\varepsilon^e - \varepsilon^i)\, R_0^2, \quad \text{at} \quad r = R_0$$

and

$$W = \frac{(\varepsilon^e - \varepsilon^i)}{2} \int_0^{2\pi} \int_{kz=0}^{2\pi} \int_0^{R_0} (\boldsymbol{E}_0^3 . \boldsymbol{E}_1^i)\, r\, dr\, dkz\, d\phi. \qquad (69)$$

Carrying out the integrations in (69) and substituting from (68) and (69) into (67), the total change in the electric energy (per unit length) due to the deformation (11 a) of the fluid cylinder is given by

$$W_1 = \frac{\pi R_0^3 E_0^2 \cos^2 wt}{2} \left\{ - \frac{[x\, \varepsilon^i - \varepsilon^e (x + m\beta)]^2\, I_m(x)\, K_m(x)}{x(\varepsilon^i I_m'(x) K_m(x) - \varepsilon^e K_m'(x) I_m(x)} + \varepsilon^e \beta^2 \right\} \qquad \ldots (70)$$

6. Characteristic Value Problem

By the use of (56),(64) and (70) for (12), the Lagrangian function is construced and moreover if we use the Lagrangian equation (13) we obtain

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$$\frac{d^2\gamma}{dt^2} - 4\pi G\rho \left(\frac{x\, I_m'(x)}{I_m(x)}\right)(I_m(x)\, K_m(x) - \tfrac{1}{2})\gamma$$

$$+ \frac{E_0^2\gamma}{\rho R_0^2}\cos^2(w\,t)\,\frac{z\, I_m'(x)}{I_m(x)}\left[\frac{[x(\alpha\varepsilon^e - \varepsilon^i) + m\beta\varepsilon^e]^2 I_m K_m}{[\varepsilon^i I_m'(x)K_m(x) - \varepsilon^e I_m K_m']} - \varepsilon^e\beta^2\right] = 0 \quad \ldots (71)$$

Equation (71) is an integro-differential equation governing the surface displacement γ of the perturbation and may be rewritten in the following brief form

$$\frac{d^2\gamma}{d\eta^2} + (b - h^2\cos^2\eta)\gamma = 0 \quad \ldots (72)$$

where η and b are defined by

$$\eta = wt,\ b = \frac{4\pi G\rho}{2}\,\frac{x\, I_m'(x)}{I_m(x)}\left[\frac{1}{2} - I_m(x)K_m(x)\right] \quad \ldots (73, 74]$$

$$h^2 = \frac{-E_0^2}{\rho R_0^2 w^2}\left(\frac{x\, I_m'(x)}{I_m(x)}\right)\left\{\frac{(x\,\varepsilon^i - \varepsilon^e(\alpha x + m\beta))^2\, I_m(x)K_m(x)}{x[\varepsilon^i I_m(x)K_m(x) - \varepsilon^e I_m(x)K_m'(x)]} - \varepsilon^e\beta^2\right\} \quad \ldots (75)$$

The second order ordinary differential equation (72) has the canonical form

$$\frac{d^2\gamma}{d\eta^2} + (a - 2q\cos 2\eta)\,\gamma = 0 \quad \ldots (76)$$

where

$$a = b - \frac{1}{2}h^2,\quad q = \frac{1}{4}h^2. \quad \ldots (77)$$

7. Limiting Cases

Under appropriate choices we can obtain the following limiting cases :

If we put $\beta = 0$ and $G = 0$ in equation (71) we recover the dispersion relation of Reynolds [19] (eqn (44)) if we neglect the surface tension and charges there and assume that fluid cylinder is ambient with vacuum medium.

If we suppose that $G = 0$, $\beta = 0$ and $w = 0$, equation (71) degenerates to that of N { symbol f }ayyar and Murty [12] (eqn (23)) if we neglect the capillary force there.

Inserting $\beta = 0$ and $G = 0$ in equation (71) we recover the characteristic equation of Abouelmagd and Nayyar [1] (eqn (2.7)) if we neglect the surface tension influence there.

Following Chandrashekhar and Fermi [5] we postulate that $\gamma \sim \exp(\sigma t)$ where σ is the growth rate, and moreover we assume here

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that $E_0 = 0$, $\beta = 0$, $\alpha = 0$, $w = 0$ and simultaneously $m = 0$; equation (72) yields

$$\sigma^2 = 4\pi\rho\, G \frac{x\, I_1(x)}{I_0(x)} ((I_0(x)\, K_0) - \tfrac{1}{2}), \qquad \ldots (78)$$

where use has been made of the relation $I_0'(x) = I_1(x)$. The dispersion relation (78) has been derived for first time by Chandrashekhar and Fermi [5] for the aim of investigating the self-gravitating fluid cylinder dispersed into a self-gravitaing vacuum. Indeed they [5] have utilized a totally different technique from that used here. Their perturbation technique [5] is mainly depends on representing the solenoidal vector fields in terms of poloidal and toroidal vector fields, and that analysis is only valid for axisymmetric perturbation mode $m = 0$ but not for those of non- axisymmetric $m \neq 0$.

If we postulate $\gamma \sim \exp(\sigma t)$ and we moreover assume that $E_0 = 0$, $m \neq 0$, $\beta = 0$, $\alpha = 0$ and $w = 0$, the characteristic equation (72) yields

$$\sigma^2 = 4\pi G\rho \frac{xI_m'(x)}{I_m(x)} (I_m(x)\, K_m(x) - \tfrac{1}{2}). \qquad \ldots (79)$$

This dispersion relation is derived and discussed by Chandrasekhar [4] on utilizing the noraml mode analysis, and did mention the correlation of such study with understanding the dynamical behaviour of the spiral arms of galaxies.

If we postulate that $\gamma \sim \exp(\sigma t)$ and we put $\beta = 0$ and $w = 0$, we obtain from (72) an electrogravitational dispersion relation of a fluid cylinder pervaded by uniform electric field $(0, 0, E_0)$ dispersed in gravitating medium penetrated by the longitudinal uniform electric field $(0, 0, \alpha E_0)$, see Radwan [18].

If we assume that $\gamma \sim \exp(\sigma t)$ and simultaneously we suppose that $w = 0$, $\alpha = 0$ and $m \geq 0$, the characteristic equation (72) degenerates to a dispersion relation of a self-gravitating fluid cylinder pervaded by $(0, 0, E_0)$ and surrounded by the varying electric field $(0, \beta R_0 r^{-1} E_0, 0)$.

Other stability criteria could be derived in the cases :

(*i*) As $G = 0, E_0 \neq 0$, $\beta \neq 0$, $\alpha = 0$, $w = 0$ and $m = 0$.

(*ii*) As $G = 0$, $E_0 \neq 0$, $\beta = 0$, $\alpha \neq 0$, $w = 0$ and $m = 0$.

(*iii*) The cases (*i*) and (*ii*) as $m \geq 0$.

8. Stability Discussions

The integro-differential equation (72) or rather its canonical form (76) is the Mathieu second order differential equation and its

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solution is given in terms of Mathieu functions. These transcendental functios (their behaviour, numerical data, ... etc.) are studied in several text books, see for example Morse and Feshback [11] and Mclachlan [10]. We have to mention here that our model of the gas-core fluid cylinder is being stable if the solution of equation (72) is periodic and this could be occurred under certain restrictions. These appropriate restrictions are depending on the relationship which correlate the parameters q and α. In numerical studies presenting q on a horizontal axis and α as vertical axis, it is found that the (q, α)-plane is classified into different categories. These categories are corresponding to stable and unstable regions bounded the characteristic curves of the Mathieu functions, see Ince [8] or/and Mclachlan [10]. Therefore we predict that the model of the gas-core fluid jet is stable if the solution of the Mathieu differential equation (72) is obtained such that the point (q, α) lies interior or/and on the boundaries of a stable domain in the stable regions and vice versa. Hence the condition for stability degenerates to the problem of the boundary regions of Mathieu function. Mclachlan /10/ gave the explicit condition

$$\left| D(0) \sin^2 \frac{\pi a}{2} \right|^{1/2} < 1 \qquad \ldots (79)$$

for stability where $D(0)$ is the Hill's determinant. However the analysis of this condition is useless and rather cumbersome because $D(0)$ is infinite.

The numerical discussions and investigations concerning the stable and unstable regions of the characteristic curves of the Mathieu functions reveal, for very small values of q, that the first unstable region is bounded by the curves

$$\alpha = 1 \pm q \qquad \ldots (80)$$

while the boundary curves of the other unstable regions which are higher than the first unstable regions are totally different from (80). However, in this respect, Morse and Feshbach [11] have found out an approximate relation for identifying the (in-) stability states. That aproximate relation is, valid only for very small values of h^2 being

$$(h^4 - 16(1-b)h^2 + 32b(1-b)) \geq 0 \qquad \ldots (81)$$

If the restrictions (81) are satisfied the model must be stable and vice versa where the equality is corresponding to the marginal (neutral) stability. One have to refer here that $|h^2|$ is very small as the periodic field frequency w is very high. The inequality (81) is quadratic in h^2 and could be expressed as

$$(h^2 - \alpha_1)(h^2 - \alpha_2) \geq 0. \qquad \ldots (82)$$

with

$$\alpha_1 = 8(1-b) - D \text{ and } \alpha_2 = 8(1-b) + D \qquad \ldots (83), (84)$$

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are the two roots of the equality in the restrictions (81) where

$$D = \{32\,(1-b)(2-3b)\}^{1/2}. \qquad \ldots (85)$$

It is found more convenient to study and write down some properties of the modified Bessel functions before carrying out the instability and oscillation investigations of the present problem.

Consider the recurrence relation (cf. Gradshteyn and Ryzhik [7], of the modified Bessel functions

$$2\,I_m'(x) = I_{m+1}(x) + I_{m-1}(x) \qquad \ldots (86)$$

$$2\,K_m'(x) = -K_{m+1}(x) - K_{m-1}(x). \qquad \ldots (87)$$

Also consider the facts, for every non-zero real value of x, that $I_m(x)$ is positively definite and monotonic increasing while $K_m(x)$ is monotonic decreasing but never negative i.e.

$$I_m(x) > 0, \quad K_m(x) > 0 \qquad \ldots (88, 89)$$

By the use of the relations (86) and (87) and the inequalities (88) and (89) we can show, for every $x \neq 0$, that

$$I_m'(x) > 0, \quad K_m'(x) < 0. \qquad \ldots (90, 91)$$

Consequently for any positive values of ε^i and ε^e and $x \neq 0$, we get

$$I_m(x)\,K_m(x) > 0 \qquad \ldots (92)$$

$$x(I_m'(x)/I_m(x)) > 0 \qquad \ldots (93)$$

$$[\varepsilon^i\,I_m'(x)\,K_m(x) - \varepsilon^e\,I_m(x)\,K_m'(x)] > 0. \qquad \ldots (94)$$

However, it is found numerically concerning the inequality (92), (see Chandrasekhar [4]), that

$$I_m(x)\,K_m(x) < \tfrac{1}{2} \text{ as } m \geq 1. \qquad \ldots (95)$$

In the axisymmetric mode of peryurbation $m = 0$, the value of the compound functions $I_0(x)\,K_0(x)$ may be larger or smaller than $\frac{1}{2}$ and that depends on the x value. Moreover it is found that

$$I_0(x)\,K_(x) \leq \tfrac{1}{2} \text{ as } \quad 1.0668 \leq x < \infty \qquad \ldots (96)$$

$$I_0(x)\,K_0(x) \geq \tfrac{1}{2} \text{ as } \quad 0 < x < 11.0668 \qquad \ldots (97)$$

In view of the foregoing relations and inequalities, using (75), we can show that

$$h^2 < 0$$

provided that

$$\{x[\varepsilon^i I_m'(x)K_m(x) - \varepsilon^e I_m(x)K_m'(x)]\,\varepsilon^e\,\beta^2\} < ([x\varepsilon^i - \varepsilon^e(\alpha x + m\,\beta)]^2\,I_m(x)K_m(x))$$

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Now, by an appeal to the inequalities (92), (93) and (95) to (98) for discussing the dispersion relation (79), we may identify the self-gravitating force influence. Following Chandrasekhar's analysis [4], it is found that the model is purely self-gravitating stable in the modes $m \geq 1$ for all $x \neq 0$ and also in the mode $m = 0$ in the domains $1.0668 \leq x < \infty$. It is only self-gravitating unstable in the axisymmetric mode $m = 0$ in the domain $0 < x < 1.0668$.

In order to determine the influences of the different acting electrodynamic forces on the present model with neglecting the periodicity of the basic electric fields, we would have to use the stability criterion

$$\frac{\rho R_0^2 \sigma^2}{E_0^2 \varepsilon^e} = \frac{x I_m'(x)}{I_m(x)} \left[\beta^2 - \frac{[x(\varepsilon^i/\varepsilon^e) - (\alpha x + m\beta)]^2 I_m(x) K_m(x)}{x[(\varepsilon^i/\varepsilon^e) I_m'(x) K_m(x) - I_m(x) K_m'(x)]} \right] \quad \ldots (99)$$

which is coming out from (71) by considering $\gamma \simeq \exp(\sigma t)$ and $G = 0$. The quantity $R_0(\rho)^{1/2}/(E_0(\varepsilon^e)^{1/2})$ has a unit of time therefore, the relation (99) is a dimensionless equaiton. In equation (99) there is no any term free from the parameters α and β. This means that the interior electric field $\boldsymbol{E}_0^i$ has no influence on the stabilty of the present problem. The influence of the axial exterior electric field $(\boldsymbol{E}_0^e)_z$ is represented by the term, including α in equation (99) as $\beta = 0$,

$$\frac{x^2[(\varepsilon^i/\varepsilon^e) - \alpha]^2 I_m'(x) K_m(x)}{I_m(x) K_m'(x) - (\varepsilon^i/\varepsilon^e) I_m'(x) K_m(x)]}. \quad \ldots (100)$$

By the use of the inequalities (89), (90), and (94), we find that the axial exterior electric field is stablizing

$$\varepsilon^i > 0, \quad \varepsilon^e > 0, \quad \alpha > 0. \quad \ldots (101)$$

This result is valied for all values of $x \neq 0$ in the axisymmetric mode $m = 0$ and also in those of non-axisymmetric $m \neq 0$.

The effect of the azimuthal exterior electric field $(\boldsymbol{E}_0^e)_\phi$ is represented by the terms, including β in equation (99) as $\alpha = 0$,

$$\frac{x I_m'(x)}{I_m(x)} \beta^2 \left[1 - \frac{m^2 I_m(x) K_m(x)}{x[(\varepsilon^i/\varepsilon^e) I_m'(x) K_m(x) - I_m(x) K_m'(x)]} \right] \quad \ldots (102)$$

In the axisymmetric mode $m = 0$, it is purely destabilizing due to its contribution $\beta^2(x I_1(x)/I_0(x))$ where we have used the relation $I_0'(x) = I_1(x)$.

In the non-axisymmetric perturbation modes $m \geq 1$, the exterior azimuthal electric field has a strong stabilizing influence if, for each non-zero real value of x, the restrictions

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$$(m^2 I_m(x)K_m(x)) \le x[(\varepsilon^i/\varepsilon^e)I_m'(x)K_m(x) - I_m(x)K_m'(x)] \quad \ldots (103)$$

are satisfied and vice versa where use has been made of the inequalities (88) to (94).

We may conclude here that the electrodynamic forces (with general variable electric field) acting on the present model are stabilizing if

$$x(\varepsilon^i I_m'(x)K_m(x) - \varepsilon^e I_m(x)K_m'(x))\beta^2 \le [x\varepsilon^i - \varepsilon^e(\alpha x + m\beta)]^2 I_m(x)K_m(x)$$

.... (104)

where the equalities are associated witht he neutral stabilities. Indeed these general restrictions are deduced without paid any atention for the magnitude of E_0. The effect of the latter will be determined later through the investigation of the integro- differential equation (72) or rather its canonical form (76) and this will be our present scope.

Now, let us return to the general characteristic Mathieu differential equation (72). Following Morse and Feshbach [20] principles with taking into accounts the relations (83) to (85) for the inequalities (81) and (82), the (in-) stability investigations and discussions should be carried out in the following different cases.

The case $0 < b < (2/3)$

In this case we have $b > 0$ and simultaneously $3b < 2$ we can show that D is real. Hence we can prove that α_1 is positive, note that if α_1 is positive, then α_2 is also positive because $\alpha_2 > \alpha_1$ as follows. If α_1 is negative, we have from (83) that $(8(1-b) - D) < 0$ and on using (85) we get $b > 1$ which is a contradiction to the present postulates that $0 < b < (2/3)$ of the our case. Therefore, α_1 must be positive and α_2 is so. Now since α_1 and α_2 are real and positive we predict, on using (98), that each of $(h^2 - \alpha_1)$ and $(h^2 - \alpha_2)$ is negative, Hence the product $(h^2 - \alpha_1)(h^2 - \alpha_2)$ is positive definite and this shows that the restrictions (81) are statisfied under the conditions $0 < b$ and $3b < 2$.

The case $(2/3) < b < 1$.

This case is restricted by the two conditions $(2/3) < b$ and $b < 1$ simultaneously. Using these together with (98) we can prove that the two roots α_1 and α_2 are complex. Consequently the stability restrictions (81) are indentically satisfied.

Summarizing the results of the foregoing cases $0 < b < (2/3)$ and $(2/3) < b < 1$, we see that the stability restrictions (81) are satisfied and hence the model at hand is stable if b is bounded by

$$0 < b < 1. \quad \ldots (105)$$

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Combining (74) and (105), we conclude that the modulating dielectric self-gravitating fluid cylinder is stable if the frequency of the periodic electric field is larger than a critical value w_c such that

$$w_c^2 = \left[4\pi G\rho \frac{xI_m'(x)}{I_m(x)} \left(\tfrac{1}{2} - I_m(x)\, K_m(x)\right) \right] > 0 \qquad \ldots (106a)$$

This restriction is independent of the amplitude of the applied electric field. Therefore, upon choosing appropriate value of W the model could be in stabilizing state. Hence an investigation of the right side in the equality (106a) is leading to find out exactly where are the domains of stability and those of instability. This is our present scope.

The equality in (106a) may be rewritten in the non-dimension form

$$\frac{w^2}{4\pi G\rho} = \frac{xI_m'(x)}{I_m(x)} \left(\tfrac{1}{2} - I_m(x)\, K_m(x)\right). \qquad \ldots (106b)$$

By the use of the inequalities (93) and (95) for this relation, we conclude that the electirc field frequency is strongly destabilizing in all non-axisymmetric perturbation modes $m \geq 1$. Also in using the inequality (93) in the axisymmetic mode $m = 0$ of perturbation and taking into account the inequalities (96) and (97) for disscusing the relation (106 b) we deduce the following. The electric field frequency w has strong stabilizing influence in the doman $0 < x \leq 1.0668$ 1.0668 while it has a destabilizing effect in the doman $1.0668 < x < \infty$ upon axisymmetirc perturbation.

Therfore, we conclude that the frequency of the electric field is stabilizing for axisymmetric mode in the doman $0 < x \leq 1.0668$ only and destablizing otherwise.

The case $b > 1$.

In this case we have $b > 1$ so $3b > 2$ and we can also show that the determinant D is being real. Moreover we can show that α_1 is negative here as follows. If α_1 is positive, using (83), we get

$$8(1-b) \geq D. \qquad \ldots (107)$$

From the view point of (85), the restrictions (107) may be rewritten as

$$\left(64(1-b)^2\right) \geq \left(32(1-b)\,(2-3b)\right) \qquad \ldots (108)$$

from which we obtain $b < 1$. The latter result is a contradiction to the supposition of the present case that $b > 1$, hence α_1 must be negative. In similar steps we can show that α_2 is positive, hence on using the inequality (98) we have

$$\left(h^2 - \alpha_2\right) < 0 \qquad \ldots (109)$$

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for all values of b (> 1). By the use or (109), the stability restricitions (82) would be satisfied only if

$$h^2 \geq \alpha_1 \qquad \ldots (110)$$

or alternatively

$$E_0^2 \geq \frac{\rho^2 R_0^2 w^2}{F_m^2(x)} [8(1-b)+D] \qquad \ldots (111)$$

with

$$F_m^2(x) = \frac{xI_m'(x)}{I_m(x)} \left\{ \frac{x\varepsilon^i - \varepsilon^e(\alpha x + m\beta)]^2 I_m(x) K_m(x)}{x[\varepsilon^i I_m'(x) K_m(x) - \varepsilon^e I_m(x) K_m'(x)]} - \varepsilon^e \beta^2 \right\} \qquad \ldots (112)$$

where use has been made of (75) and (83).

The case $b < 0$

This case in which b is assumed to be negative looks like the previous case in which $b > 1$ where we can show that

$$\alpha_1 = -\text{ve} \text{ and } \alpha_2 = +\text{ve} \qquad \ldots (113)\ (114)$$

Consequently the stability restrictions (82) are satisfied under the validity of the inequalities (109) and (110).

Therefore, we conclude that for the cases $b < 0$ and $b > 1$, the dielectric fluid cylinder will be stable if there exist a critical value E_0^C of the electirc field intensity E_0 such that $E_0 \geq E_0^C$ where

$$E_c = \frac{wR_0\rho}{F_m(x)} (8(1-b)+D)^{1/2}. \qquad \ldots (115)$$

It is worthwhile to mention here that the quantity $(\varepsilon^e E_0^2/(\rho R_0^2))^{-1/2}$ has a unit of "time" and therefore we may formulate the characteristic relation (115) in a dimensionless form. Thence by giving appropriate values for the occurred different parameters in regular steps of the wavenumber x, we could find out the critical value of E_p i.e. E_0^c above which the instability character of the model is completely suppressed and stability then arises and sets in.

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SOME COUNTEREXAMPLES IN FIXED POINT THEORY

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ABSTRACT

We give a few counterexamples in Fixed Point Theory for multivalued mappings.

In this short paper we present a few counterexamples of various nature in the area of Fixed Point Theory for multivalued mappings.

They are not directed to an audience of experts in the field, but they are devoted to an audience of neophytes of Fixed Point Theory; neophytes are above all rich of little experience.

Although the counterexamples presented here are very elementary, we think they can be more useful than laborious but "fancy" fixed point Theorems.

In the following we will label :

- H a Hilbert space.
- E a Banach space with norm $\| \ \|$.
- X a metric space with distance d.
- 2^X the non empty subsets of X.
- K a subset of X.
- $CB(X)$ the family of nonempty bounded closed subsets of X.
- $KC(E)$ the family of nonempty compact convex subsets of E.
- For $A, B \in CB(X)$, $D(A, B) := \max (\sup_{b \in B} d(b, A), (\sup_{b \in A} d(a, B))$, where $d(X, Y) := \inf_{y \in Y} d(x, y)$, is the Hausdorff distance from A to B.
- f a single-value map from K to X.
- T a multivalued map from K to 2^X.
- $Fix(T)$ the set of fixed points of T, $Fix(T) := \{z \in K : z \in Tz\}$.
- For $A, B \in CB(X)$, $\delta(A, B) := \sup \{d(a, b) : a \in A, b \in B\}$, $\delta(x, A) := \delta(\{x\}, A)$.

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COUNTEREXAMPLES 1, 2 AND 3.

In [1] Caristi states (and it could not be said better): "An inwardness condition is one which assert that, in some sense, points from the domain are mapped "toward" the domain . . . Possibly the weakest of the inwardness conditions, the Leray- Schauder boundary condition, is the assumption that T maps points x of ∂K anywhere except to the outward part of the ray originating at some interior point of K and passing through x, i.e.

$\exists\, w \in K^0$ such that " $x \in \partial K$, $f(x) - w = m(x - w)$ implies $m \leq 1$. ... (1)

If $x \in K$ we define the inward set, $I_K(x)$, as follows :

$$I_K(x) := \{x + c(u - x) \in E : u \in K \text{ and } c \geq 1\}.$$

A mapping $f : K \to E$ is said to be inward if $f(x) \in I_K(x)$ $\forall\, x \in K$.

We say that f is weakly inward in case $f(x) \in I_K(x)$ $\forall$ $x \in K$.

The proof of the fact that the Leray-Schauder condtion is weaker than the inwardness condition defined above, can be found for instance in [2] in the case K is convex.

The following counterexample shows that the converse is not true :

COUNTEREXAMPLE 1. Let $E = R^2$ with the euclidean norm

$$K := \{(a, b) \in E : a^2 + b^2 \leq 1\},\ f(a, b) := (a, b + |a|).$$

We show that f satisfies (1) with $w = 0$, i.e., $f(x) = mx$ implies $m \leq 1$.

Let $x = (a, b) \in \partial K$. Then $x = (a, \pm (1 - a^2)^{1/2})$ and $f(x) = (a, \pm (1 - a^2)^{1/2} + |a|)$.

Moreover let $f(x) = mx$, that is

$$\begin{cases} a = ma & (2) \\ \pm (1 - a^2)^{1/2} + |a| = m(\pm (1 - a^2)^{1/2}) & (3) \end{cases}$$

Now, if $a \neq 0$, furnish $m = 1$ and then (3) yields $a = 0$, a contradiction ! So $f(x) = mx$ only if $x = (0, \pm 1)$ and in this case (3) implies $m = 1$.

On the other had, one can see immediately that f is not weakly inward.

Some extensions of the (Weak) inwardness and Leray-Schauder condtions are useful also in the search for fixed points for multivalued mappings

(Results and references can be found for example in Deimling [2], [3]).

For example, if $T : K \to 2^E$ is a multivalued mapping, a suitable reformulation of (1) is

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$\exists\, w \in K^0$ such that $\forall\, x \in \partial K,\ w + m(x - w) \in Tx$ implies $m \leq 1$ ... (4)

and a suitable reformulation of inwardness is

$$T x \cap I_K(x) \neq \phi \quad \forall\, x \in K. \qquad \ldots (5)$$

However, while in the univoque case (5) implies (4) whenever K is convex, but in the multivalued case this is no longer true :

COUNTEREXAMPLE 2.

Take E, K as in counterexample 1.

$$T(a, b) := (a, [b, b + 1]).$$

Then $(a, b) \in T(a, b)$ for $(a, b) \in K$, so T satisfies (5). We show that T does not satisfy (4).

Let $w \in K^0$, $w = (a_0, b_0)$. Let $x = x(w) = (a_0, (1 - a_0^2)^{1/2}) \in \partial k$, $z = (a_0, (1 - a_0^2)^{1/2} + 1)$, $m = 1 + ((1 - a_0^2)^{1/2} - b_0)^{-1} > 1$. Then $w + m(x - w) \in Tx$ and this means (4) is not satisfied.

REMARK. We observe that the map of Counterexample 2 verifies a wak Leray-Schauder condtion : $\exists\, w \in K^0$ such that $\forall\, x \in \partial K$, $w + m(x - w) \in Tx$ implies $\exists\, c \leq 1$ such that $w + c(x - w) \in Tx$. The last condtion was introduced by De Pascale-Guzzardi in [4] and in our opinion it is the true formulation of the Leray-Schauder condtion in the context of multivalued maps with convex values.

A similar situation occurs for nonexpansive and pseudocontractive maps : it is immediate to see that, in the univoque case, nonexpansivity implies pseudocountractivity, i.e. $\| f(x) - f(y) \| \leq \| x - y \|$ implies

$$\| x - y \| \leq \| (1 + r)(x - y) - r(f(x) - f(y)) \| \quad \forall\, x, y \in K, r > 0.$$

In the multivalued case it is no longer true :

COUNTEREXAMPLE 3.

Let $E = R$, $K = E$, $Tx := [x, x + 1]$. T is nonexpansive (i.e. $D(Tx, Ty) \leq \| x - y \|$ but not pseudocontractive (i.e. $\| x - y \| \leq \| (1 + r)(x - y) - r(u - v) \|$ $\forall\, x, y \in E,\ u \in Tx,\ v \in Ty, r > 0$ is not satisfied).

COUNTEREXAMPLES 4 AND 5

In the last years, the interest in optimization theory for the multivalued maps T satisfying Fix $(T) = \{z\}$ and $\{z\} = Tz$, has prompted a corresponding interest in Fixed Point Theory, since in [5] it has been shown that the maximization of a multivalued map T with respect to a cone, which subsumes ordinary and Pareto optimization, is equivalent to a fixed point problem of determining z such that $Tz = \{z\}$.

With this purpose, a sufficient condition that ensures $Fix(T) = \{z\}$ and $Tz = \{z\}$ for a multivalued map $T : X \to CB(X)$ is, for example

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$$\delta(Tx, Ty) \leq q \max\{d(x, y), \delta(x, Tx), \delta(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}$$

with $0 < q < 1$ ([6]).

On the other hand, if the multivalued mapping T satisfies

$$\delta(Tx, Ty) \leq Ad(x, Tx) + Bd(y, Ty) + cd(x, y) \quad \ldots (6)$$

with $A, B \in]0, \infty[$ and $c \in]0, 1[$

or

$$\delta(Tx, Ty) \leq a\delta(x, Tx) + b\delta(y, Ty) + cd(x, y) \quad \ldots (7)$$

then $Fix(T) \neq \phi$ obviously implies $Fix(T) = \{z\}$ and $Tz = \{z\}$.

A sufficient condition which ensures $Fix(T) \neq \phi$ is, for example

$$D(Tx, Ty) \leq q \max\{d(x, y), \tfrac{1}{2}[d(x, Tx) + d(y, Ty)], \tfrac{1}{2}[d(x, Ty) + d(y, Tx)]\}$$

with $0 < q < 1$ [7]. ... (8)

However we note that the map $T : N \to N$ defined by $T0 = T1 = 0$, $Tn := \{0, 1\}$, $n \geq 2$, satisfies both (6) and (7) with $A = 10$, $B = 10$, $a = 0.4$, $b = 0.5$, $c = 0.9$ but it does not satisfy (8), although Fix $(T) \neq 0$. one could suspect that the condition (6) or (7) always implies $Fix\ (T) \neq \phi$. This is not true, as the following example shows :

COUNTEREXAMPLE 4.

Let $X = N$, $T0 := 1$, $Tn := \{0, 1\}$ for $n \geq 2$. Then (6) and (7) both are satisfied with $A = 20$, $B = 20$, $a = 0.4$, $b = 0.5$, $c = 0.9$. Nevertheless Fix $(T) = \phi$.

At this point, we recall that interesting results are known for the weak contraction single valued mappings (i.e. $d(f(x), f(y)) < d(x, y)$). Such contractions were studied first by Edelstein in [8].

In the case of a multivalued map T, taken into account that (6) and (7) are not sufficient to guarantee $Fix\ (T) \neq \phi$, not even if the domain of T is a compact set, one can suspect that the following condition is sufficiently strong :

$$\delta(Tx, Ty) < d(x, y) \text{ for } x \neq y. \quad \ldots (9)$$

Actually, such a condition is too strong, in the sense that nearly always do not exist "authentic" multivalued maps that satisfy (9) as shown in the following lemma :

LEMMA Let X be a metric space without isolated points. Let $T : X \to 2^X$ be a multivalued mapping satisfying (9). Then $T = f$ is a single valued map.

PROOF. Let $x \in X$ and $y_n \in B(x, 1/n)$, $y_n \neq x$. Then (9) implies

$$\delta(Ty_n, Tx) \to 0 \text{ for } n \to \infty \quad \ldots (10)$$

and from this its follows that Tx is a singleton. Indeed, if there exist $z, w \in Tx$ with $z \neq w$, we put $r = \dfrac{1}{2} d(z, w)$ and we show that

$$\delta(Ty_n, Tx) \geq r \text{ for each } n \quad \ldots (11)$$

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contradicting (10). If fact, if $\forall\ v \in Ty_n$ it results $d(z, v) \geq r$, then (11) is obviously true. On the contrary, if there exists $v \in Ty_n$ such that $d(z, v) < r$ then $2r = d(z, w) \leq d(z, v) + d(v, w) < r + d(v, w)$, i.e. $d(v, w) > r$ and this still yields (11) since $(Ty_n, Tx) \geq d(v, w)$.

On the contrary, if the metric space X has isolated points, there exist multivalued mappings T satisfying (9) and in such a case $Fix(T)$ is a singleton ($Fix(T) = \{z\}$) of course. But that which is not verified in general is $Tz = \{z\}$ as the following counterexample shows :

COUNTEREXAMPLE 5.

$$X = \{0, 1, 2/3\} \cup \{2/3 - \sum_{k=1}^{n} 10^{-k},\ n \geq 1\}.\ T : X \to 2^X,\ T0 := \{0, 1\},$$

$$T1 := 2/3) := 2/3 - 10^{-1},\ T(2/3 - \sum_{k=1}^{n} 10^{-k}) = 2/3 - \sum_{k=1}^{n+1} 10^{-k}.$$

Then T satisfies (7), $Fix(T) = \{0\}$ but $T0 \neq \{0\}$.

COUNTEREXAMPLE 6

Finally, let H be a Hilbert space and K a closed convex subset of H. For a map $T : K \to KC(H)$ we define $\forall\ x \in K$

$$\hat{T}x := \{y \in Tx : d(y, K) = d(Tx, K)\},$$

the Fan's best approximation from Tx to K.

One can see that the convexity of Tx implies the convexity of $\hat{T}x$ and $P_K\hat{T}x$, is projection on K[9].

But without the assumption of convexity of Tx, the reader is invited to find examples of mappings such that T is nonexpansive while $\hat{T}$ is not.

In [8] was an open question if the nonexpansivity of $T : K \to KC(H)$ implies the nonexpansivity of $\hat{T}$. The following counterexample, due to *H.K.Xu*, negatively answers to such question :

COUNTEREXAMPLE 6.

$H = R^2$ with the euclidean norm. $K =$ triangle with vertices $D(\frac{1}{2}, 0)$, $A(1, 0)$ and $C(\frac{1}{2}, \frac{1}{2})$. Define $T : K \to KC(H)$ as follows. Let $z = (x, y)$ be any point in K and let z' be the symmetric point of z with respect to the segment CD. Let P be the projection of z' onto the x-axis. Then we define

$Tz :=$ the segment BP linking B and P.

Then if $z = (x, y)$ and $z_1 = (x_1, y_1)$, we have $D(Tz, Tz_1) = |P - P'| \leq \|z - z_1\|$,

i.e. T is nonexpansive. It is also easily seen that

$$Tz = \begin{cases} P & \text{if } z \neq A \\ \text{or } H & z = A. \end{cases}$$

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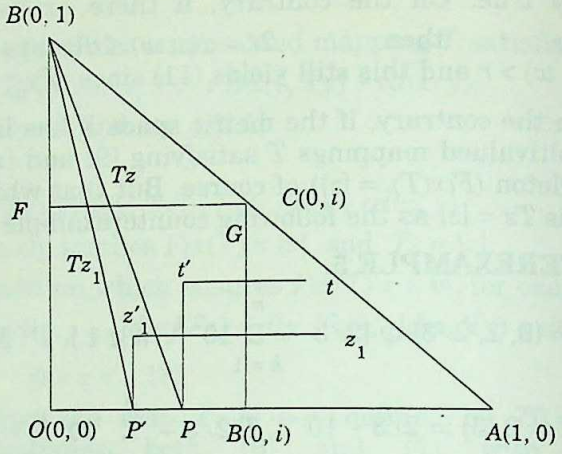
It follows that for G on the open segment CD,



$D(\hat{T}G, \hat{T}A) = \sup_{z \in 0F} d(z, D) = d(F, D) = d(C, A), d(G, A)$, so T is not nonexpansive.

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AN INTRODUCTION TO THE LIE-SANTILLI ISOTOPIC THEORY

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ABSTRACT

Lie's theory in its current formulation is linear, local and canonical. As such, it is not applicable to a growing number of nonlinear, nonlocal and noncanonical systems which have recently emerged in particle physics, superconductivity, astrophysics and other fields. In this paper, which is written by a physicist for mathematicians, we review and develop a generalization of Lie's theory proposed by the Italian-American physicst R.M. Santilli back in 1978 when at the Department of Mathematics of Harvard University and today called *Lie-Santilli isotheory*. The latter theory is based on the so-called isotopies which are nonlinear, nonlocal and noncanonical maps of any given linear, local and canonical theory capable of reconstructing linearity, locality and canonicity in certain generalized spaces and fields. The emerging Lie-Santilli isotheory is remarkable because it preserves the abstract axioms of Lie's theory while being applicable to nonlinear, nonlocal and noncanonical systems. After reviewing the foundations of the Lie-Santilli isoalgebras and isogroups, and introducing seemingly novel advances in their interconnections, we show that the Lie-santilli isotheory provides the invariance of all infinitely possible, non linear, nonlocal and noncanonical deformations of conventional Euclidean, Minkowskian or Riemannian invariants. We also show that the nonlinear, non local and noncanonical symmetry transformations of deformed generalized invariants are easily computable from the linear, local and canonical symmetry transforms of the original invariants and the given deformations. We then briefly indicate a number of applications of the isotheory in various fields. Numerous rather intriguing and open mathematical problems are indicated during the course of the presentation.

1. INTRODUCTION

1. A. Limitations of Lie's theory. As it is well known, *Lie's theory* has permitted outstanding achievements in various disciplines. Nevertheless, in its current conception [30] and realization (see, e.g.,

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[15]), Lie's theory is *linear, local- differential and canonical-Hamiltonian*. As such, it possesses clear limitations.

An illustration is provided by the historical distinction introduced by Lagrange [29], Hamilton [14] and others between the *exterior dynamical problems* in vacuum and the *interior effectively* approximated as being point-like while moving within the homogeneous and isotropic vacuum under action-at-a-distance interactions (such as a space-ship in a stationary orbit around Earth). The point-like character of particles permits the validity of conventional local-diffrential topologies (i.e., the Zeeman topology in special relativity); the homogeneity and isotropy of space then allow the exact validity of the geometries underlying Lie's theory (such as the Riemannian geometry); and the action-at-a-distance interactions assures their representation via a potential with consequential canonical character.

Interrior problems consists of extended, and therefore deformable particles moving within inhomogeneous and anisotropic physical media, with action-at-a-distance as well as contactresistive interactions (such as a space-ship during re-entry in Earth's atomosphere). In the latter case the forces are of local- differential type (e.g., potential forces acting on the center- of-mass of the particle) as well as of nonlocal integral type (e.g., requiring an integral over the surface of the body), thus rendering inapplicable conventional local-differential topologies; the inhomogeneity and anisotropy of the medium imply the inapplicability of conventional geometries for their quantitative treatment; while contact- resistive interactions violate Helmholz's conditions for the existence of a potential (the *conditions of variantional selfadjointness* [49]), thus implying the noncanonical character of interior systems.

We can therefore say that Lie's theory in its conventional linear, local and canonical formulation is *exactly valid* for all exterior dynamical problems, while it is inapplicable (ant not "violated") for the more general interior dynamical problems on topological, geometrical, analytic and other grounds.

1. B. The need for a suitable generalizaion of Lie's theory. Lie's theory is currently applied to nonlinear, nonlocal and noncanonical systems via their simplifications into more treatable forms, e.g., via the expansion of nonlocal-integral terms into power series in the velocities and then the transformation of the system into a coordinate frame in which it admit a Hamiltonian via the Lie-Koening Theorem [49].

At times, however, nonlinear, nonlocal and nonhamiltonian systems cannot be consitently reduced or transformed into linear, local and Hamiltonian ones. An illustration exists in gravitation. The distinction between exterior and interior gravitational problems was in full use in the early part of this century (see, e.g., Schwartzschild's two papers, the first celebrated paper [72], on the exterior problem and

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the second little known paper [73] on the interior problem). The distinction was then kept in early well written treatises in the field (see, e.g., [4], [38]). The distinction was then progressively abandoned up to the current treatment of all gravitational problems, whether interior or exterior, via the same local-differential Riemannian geometry.

The above trend was based on the belief that interior dynamical problems within physical media can be effectively reduced to a collection of exterior problems in vacuum (e.g., the reduction of a space-ship during re-entry in our atmosphere to its elementary constituents moving in vacuum).

It is important for this paper to know that the above reduction is mathematically impossible. For instance, the so-called *No-Reduction Theorems* [54] prohibit the reduction of a microscopic interior system (such as satellite during re-entry) *with a monotonically decreasing angular momentum*, to a finite collection of elementary particles each one with *a conserved angular momentum*, and viceversa.

On geometrical grounds, gravitational collapse and other interior gravitational problems are not composed of ideal points, but instead of a large number of extended and hyperdense particles (such as protons, neutrons and other particles) in conditions of total mutual penetration, as well as of compression in large numbers into small regions of space. This implies the emergence of a structure which is arbitrarily nonlinear (in coordinates and velocities), nonlocal-integral (in various quantities) and non- hamiltonian (variationally nonselfadjoint).

Additional insufficiencies of the current formulation of Lie's theory and of its underlying geometries exist for the characterization of antimatter, e.g., because of the lack of a suitable (e.g., antiautomorphic) map which permits the characterization of antimatter, first, at the classical astrophysical level, and then at the level of its elementary constituents.

Similar occurences have recently emerged in astrophysics, superconductivity, theoretical biology and other disciplines. These occurrences establish the need for a generalization of the conventional Lie theory which is directly applicable (i.e., applicable without approximation or transformations) to nonlinear, integro-differential and variationally nonselfadjoint equations for the characterization of matter, and then possesses a suitable antiautomorphic map for the effective characteriszation of antimatter.

1.C. Santilli's isotopies of Lie's theory. In a seminal memoir [47] written in 1978 when at the Department of Mathematics of Harvard University under support from the U.S. Department of Energy, the Italian-American scholar Ruggero Maria Santilli proposed a step-by-step generalization of the conventional Lie theory specifically conceived for nonlinear, integro-differential and noncanonical equations. The generalized theory was subsequently studied by Santilli in ref.s ([48]-[71]), as well as by a number of a mathematicians and theoreticians, and it is today called Lie-Santilli isotopic theory or isotheory (see papers [1], [2], [8], [11], [12], [16]-[23], [25], [32], [33],

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[35]-[37], [40]-[43], monographs [3], [24], [31], [74] and additional references quoted therein).

A main characteristic of the Lie-Santilli isotheory, which distinguishes it from all other possible generalizations, is its "isotopic" character intended (from the Greek meaning of the word) as the capability of preserving the original Lie axioms. More specifically, Santilli's isotopies are maps of any given linear, local and canonical structure into its most general possible nonlinear, nonlocal and noncanonical forms which are capable of reconstructing linearity, locality and canonicity in certain generalized isospaces and isofields within a fixed system of local coordinates.

The latter property is remarkable, mathematically and physically, in as much as it permits the preservation of the abstract Lie theory and the transition from exterior to interior problems via a more general *realization* of the same theory.

Another main characteristic of the Lie-Santilli isotheory is that of admitting a novel antiautomorphic map, called *isoduality*, which has resulted to be effective for the characterization of antimatter at the classical as well as operator levels.

It should be indicated that Santilli [47] submitted his isotopic theory as a particular case of a yet more general theory today called *Santilli's lie-admissble theory or Lie-Santilli genotopic theory* where the term genotopic is used (in its Greek meaning) to "induce configuration", and interpreted in the sense of violating the original Lie axioms, but inducing covering Lie-admissible axioms.

This paper is written by a theoretical physicist for mathematicians and it is soley devoted to the Lie-Santilli isotheory. A study of the broader Lie-Santilli genotheory is contemplated as a future work. In Sect. 2 we outline the methodological foundations of the theory. The isotopies of Lie's theory are presented in Sect. 3 jointly with new developments, such as a study of the transition from the Lie-Santilli isogroups to the corresponding isoalgebras. As an illustration of the capabilities of the isotheory, we prove its "direct universality" in gravitation, that is, the achievement of the symmetries of all possible gravitational metrics (universality), directly in the frame of the experimenter (direct universality). A number of fundamental open mathematical problems will be identified during the course of our analysis.

A comprehensive mathematical presentation of the Lie-Santilli isotheory up to 1992 is available in monograph [74]. A historical perspective is available in monograph [31]. Recent mathematical studies on isomanifolds (today called *Tsagas isomanifolds*) have been conducted in ref. [75] which is a topological complement of the algebraic studies of this paper.

2. Isotopies and Isodualities of Numbers, Fields, Differential Calculus, Metric Spaces, Differential Geometries, Functional Analysis, Classical and Quantum Mechanics.

Lie's theory is the embodiment of the virtual entirety of contemporary mathematics by encompassing : the theory of numbers, differential and exterior calculus, vector and metric spaces functional analysis' and others. Santilli's isotopies of Lie's theory require the isotopic lifting of all these mathematical methods. In this section we

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shall indicate the basic isotopies and isodualities which are necessary for a correct formulation of the Lie-Santilli isotheory.

2.A. Isotopies and isodualities of the unit. The fundamental isotopies from which all others can be uniquely derived are the unit I of the current formulation of Lie's theory into a quantity $\hat{I}$ of the same dimension of I, but with unrestricted functional dependence in the local coordinates x, their derivatives of arbitary order $\dot{x}, \ddot{x}, ...$as well as any needed addtional quantity [47], [49b], [61a]

$$I \to \hat{I} = \hat{I}(x, \dot{x}, \ddot{x}, ...). \qquad ... (2.1)$$

The *isotopies* occur when $\hat{I}$ preserves all the topological characteristics of I, such as nowhere-degeneracy, real-valuedness and positive-definiteness.

Once the unit is generalized, there is the natural emergence of the map [52], [53], [61a]

$$\hat{I} \to \hat{I}^d = -\hat{I}, \qquad ... (2.2)$$

called *isoduality* which provides an antiautomorphic image of all formulations based on $\hat{I}$.

The above liftings were classified by this author [22] into :

Class I (generalized units that are smooth, bounded, nondegenerate, Hermitean and positive-definite, characterizing the isotopies properly speaking);

Class II (the same as class I although $\hat{I}$ is negative-definite, characterizing isodulaties)

Class III (the union of Class I and II)

Class IV (singular isounits); and

Class V (unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices, etc.).

All isotopic structures identified below also admit the same classifcation which will be omitted for brevity. In this paper we shall generally study isotopies of Classes I and II, at times treated in a unified way via those of Class III whenever no ambiguity arises. Santilli's isotopies of Class IV and V are vastly unexplored at this writing.

2.B. Isotopies and isodualities of fields. Lie's theory is constructed over ordinary fields $F(a, +, \times)$ hereon assumed to be of characteristic zero (the fields of real $\mathcal{R}$, complex C and quaternionic numbers Q) with generic elements a, addition $a_1 + a_2$ multiplication $a_1 a_2 := a_1 \times a_2$, additive unit $0, a + 0 = 0 + a \equiv a$, and multiplicative unit $I, a \times I = I \times a \equiv a, \forall\, a, a_1, a_2 \in F$.

The Lie-Santilli isotheory is based on a generalization of the very notion of numbers and, consequently of fields first introduced by Santilli at the *Conference on Differential Geometric Methods in Mathematical Physics* held in Clausthal, Germany in 1980. A first

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rudimentary treatment appeared in Santilli's joint paper with the (mathematician) H.C. Myung [39] of 1982. Comprehensive studies were then conducted by Santilli in the following years (see paper [59] for a mathematical presentation and monographs [61] for extensive physical applications).

Consider a Class I lifting of the unit I of F, $I \to \hat{I}$ with $\hat{I}$ being outside the original set, $\hat{I} \notin F$. In order for $\hat{I}$ to be the left and right unit of the new theory, it is necessary to lift the conventional associative multiplication ab into the so-called *isomultiplication* [47]

$$ab := a \times b \Rightarrow a * b := a \times T \times a = a\, T\, b,\ T = \text{fixed}. \qquad \ldots (2.3)$$

where the quantity T is called the isotopic element. Whenever $\hat{I} = T^{-1}$, $\hat{I}$ is the correct left and right unit of the theory, $\hat{I} * a = a * \hat{I} = a$, $\forall\, a \in F$, in which case (only) $\hat{I}$ is called the *isounit*. In turn, the liftings $I \to \hat{I}$ and $\times \to *$, imply the generalization of fields into the Class I structure

$$\hat{F}_I = \{\hat{a}, +, *)\ |\hat{a} = a\hat{I};\ a = n, c, q \in F; \times \to * = \times\, T \times;\ \hat{I} = T^{-1}\}, \qquad \ldots (2.4)$$

called *isofields*, with elements $\hat{a} \in \hat{F}$ called *isonumbers* [59].

All conventional operations among numbers are evidently generalized in the transition from numbers to isonumbers. In fact, we have

$$a + b \to \hat{a} + \hat{b} = (a + b)\,\hat{I};\quad a_1 \times a_2 \to \hat{a}_1 * \hat{a}_2 = \hat{a}_1\, T\, \hat{a}_2 = (\hat{a}_1\, a_2)\hat{I};$$

$$a^{-1} \to \hat{a}^{\hat{I}} = a^{-1}\hat{I};\quad a/b = c \to \hat{a}/\hat{b} = \hat{c},\ \hat{c} = c\hat{I};\quad a^{1/2} \to \hat{a}^{\hat{1}/2} = a^{1/2}\,\hat{I}^{1/2};$$

etc. Thus, conventional squares $a^2 = aa$ have no meaning under isotopy and must be lifted into the *isosquare* $\hat{a}^{\hat{2}} = \hat{a} * \hat{a}$. The *isonorm* is

$$\hat{|}\hat{a}\hat{|} = (\bar{a}a)^{1/2}\,\hat{I} = |a|\hat{I} \in \hat{F}, \qquad \ldots (2.5)$$

where $\bar{a}$ denotes the conventional conjugation in F and $|a|$ the conventional norm. note that the *isonorm is positive-definite* (for isofields of Class I), as a necessary condition for isotopies.

The isotopic character of the lifting $I \to \hat{I}$ is confirmed by the fact that the isounit $\hat{I}$ verifies all axioms of I,

$$\hat{I} * \hat{I} * \ldots * \hat{I} \equiv \hat{I},\quad \hat{I}/\hat{I} \equiv \hat{I},\quad \hat{I}^{\hat{1}/2} \equiv \hat{I},\ \text{etc.}$$

The *isodual isofields* are the antihomomorphic image of $\hat{F}(\hat{a}, +, *)$ induced by the map $\hat{I} \to \hat{I}^d = -\hat{I}$ and are given by the Class II structures

$$\hat{F}^d_{II} = \{(\hat{a}^d, +, *^d)\ |\hat{a}^d = \hat{a}\hat{I}^d;\ a = n, c, q \in F; * \to *^d$$

$$= \times\, T^d \times,\ T^d = -T,\ \hat{I}^d = -\hat{I}\}, \qquad \ldots (2.6)$$

in which the elements $\hat{a}^d = \bar{a}\hat{I}^d$ are called *isodual isonumbers*. For real numbers we have $n^d = -n$, for complex numbers we have $c^d = -\bar{c}$, where $\bar{c}$ is the ordinary complex conjugate, and for quaternions in matrix representation we have $q^d = -q^\dagger$, where $\dagger$ is the Hermitean

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conjugate. Noe That the conjugation of a complex number is $(n+ixm)^d = n^d + i^d x^d m^d = -n + (-\bar{\imath})(-x)(-m) = -n + im$. The isodual isosum is given by $\hat{a}^d + \hat{b}^d = (\bar{a} + \bar{b})\hat{I}^d$, while for isodual isomultiplication, we have

$$\hat{a}^d *^d \hat{b}^d = \hat{a}^d T^d \hat{b}^d = -\hat{a}^d T \hat{b}^d = (\bar{a}\,\bar{b})\hat{I}^d.$$

An important property is that *the norm of isodual isofields is negative-definite* because it is characterized by

$$\hat{\uparrow}\hat{a}^d\hat{\uparrow}^d = |\bar{a}|\hat{I}^d = -\hat{\uparrow}\hat{a}\hat{\uparrow}. \qquad \ldots (2.7)$$

The letter property has nontrivial implications. For instance, it implies that *physical quantities defined on an isodual isofield, such as time, energy, etc.., are negative-definite*. For these reasons, isodual theories provide a novel and interiguing characterization of antimatter (see later on in this section the equivalence of isoduality and charge conjugation) [61].

Note also that, as a *necessary condition for isotopies (isodualities) all isofields* $\hat{F}_I(\hat{a}, +, *)$ *(isodual isofields* $\hat{F}^d_{II}(\hat{a}^d, +, *^d)$*) are isomorphic (antiisomorphic) to the original field* $F(a, +, *)$. The reader should be aware that the distinction between real, complex and quaternionic numbers is lost under isotopies because all possible numbers are unified by the isoreals owing the freedom in a generalized unit [26].

Recall that the set of imaginary numbers does not constitute a field, evidently because not closed under the multiplication. On the contrary, Santill's isofields $\hat{F}$ with isounit $\hat{I} = i$ do indeed verify the axioms for a field as one can readily verify. Note that the imaginary unit is *isoselfdual*, i.e., invariant under isoduality $i^d = -\bar{\imath} \equiv i$.

We also recall [59] that the lifting $a \to \hat{a} = a\hat{I}$ is *necessary* for $\hat{F}_I(\hat{a}, +, *)$ the preserve the axioms of $F(a, +, *)$ whenver the isounit $\hat{I}$ is not an element of the original field. On the contrary, when $\hat{I} \in F(a, +, *)$ there is no need to lift the numbers and we shall write $\hat{F}_I(a, +, *)$. In physical applications, the isounit is generally outside the original field and actually possesses a nonlinear as well as integral dependence on the local variables and their derivatives. This implies that the "numbers" used in the Lie-Santilli isotheory generally have an *integral* structure.

As an example, the isounit used by Animalu [1] for the representation of the Cooper pair in superconductivity is given by

$$\hat{I} = Ie^{t\,N\int d^3 \times \psi^\dagger(r)\,\phi(r)}, \qquad \ldots (2.8)$$

where t represents time, N is a positive real constant, and ψ and ϕ are the wave functions of the two electrons of the Cooper pair. *Annimalu's isounit* (2.8) therefore represents the nonlocal- integral contributions due to the wave overlapping of the two electrons in the

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Cooper pairs which contributions, since they are of contact type, are variationally nonself adjoint and cannot be represented with the Hamiltonian.

In particular, Animalu has shown that the lifting of the conventional Coulomb interactions characterized by isounit (2.8) produces an attraction among the *identical* electrons of the Cooper pair, as experimentally established in superconductivity. Note that when the overlapping of the wavepackets is no longer appreciable (e.g., at large mutual distances), the integral in the exponent of (2.8) is null and the isounit $\hat{I}$ recovers the conventional unit I.

Conventional fields $F(a, +, \times)$ are used for large distances among the electrons, while isofields $\hat{F}(\hat{a}, +, *)$ with isounit (2.8) are used when the wave-overlapping of the electrons is appreciable. Other examples of isounits will be provided later on.

We also recall Santilli's [59] still more general *genofields*, characterized first by an isotopy of conventional fields, and then by an ordering of the isomultiplications, one to the right $\hat{a} > \hat{b} = \hat{A} \times R \times \hat{b}$ and one to the left $\hat{a} < \hat{b} = \hat{a} \times S \times \hat{b}$, $\hat{a} > \hat{b} \neq \hat{a} < \hat{b}$, $R \neq S$. The important property is that all abstract axioms of a field are verified per each ordered isomultiplication thus yielding a generalization of isofields which is at the foundation of Santilli's Lie-admissible Theory.'

We finally recall Santilli's [59] still more general liftings characterized by the generalization of the sum $+$ and related additive unit 0, e.g., $+ \to \hat{+} = K +, 0 = K \neq 0, K \in F(a \hat{+} b = a + K + b)$ called *pseudo isotopies*, which do not preserve the axioms of a field (e.g., closure under the distributive law is not verified under the conventional $\times$ or isotopic $*$ multiplication and the addition $\hat{+}$). Thus, *pseudoisofields are not fields. For these and other reasons (e.g., the general divergence of the exponentiation), physical applications are restricted to iso-and geno-fields, while the pseudoiso-and pseudogeno-fields* have a mere mathematical interest at this writing. The care needed in inspecting and appraising the Lie- Santilli isotheory can be pointed out from these introductory lines. In fact, familiar statements such as "two multiplied by two equals four" are correct for the conventional Lie theory, but they have no mathematical meaning for the Lie-Santilli isotheory because they lack the identification of the assumed unit and multiplication. In fact, for $\hat{I} = 3$, $\hat{2} * \hat{2} = 12$. Similarly, care must be expressed before claiming that a number is *prime* or not. Infact, Santilli [59] has shown that non-prime numbers can become prime under a proper selection of the unit.

Santilli's theory of isonumbers is today sufficiently well known, and includes the lifting of all conventional numbers (real, complex and quaternionic numbers, plus the isotopies of octonions [59]) into the following four classes used in this paper : (A) *ordinary numbers* with unit 1; (B) *isonumbers* with isounits of Class I, $\hat{I} > 0$; (C) *isodual*

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numbers with isodual unit $I^d = -1$; (D) *isodual isonumbers* with isodual isounits of Class II, $\hat{I}^d < 0$. In this paper we shall therefore have *four* different types of real numbers, complex numbers and quaternions, excluding generalizations of Classes IV and V.

Despite the above advances, studies on the isonumber theory are their initiation and so much remains to be done. To begin, the entire conventional number theory (including all familiar theorems on factorization etc.) can be subjected to an isotopic lifting of Class I. Moreover, we have the birth of new numbers without counterpart in the current number theory, such as the isonumbers of Class IV (with singular isounits) and Class V (with distributions or discountinuous functions as isounits). All the above lifting then admit antiautomorphic images under isoduality which are absent in the conventional number theory. In turn, all the preceding generalizations can be subjected to a further enlargement via the differentiation of the multiplications to the right and to the left, and then yet more general formulations via the multivalued hyperstructures.

One can begin to understand the vastity of the Lie-Santilli isotheory as compared to the conventional formulation of Lie's theory by nothing that the above hierarchy of fields implies a corresponding hierarchy of Lie-isotopic theories.

2.C. Isotopies and isodualities of the differential calculus. The next important mathematical discovery by Santilli is an axiom-preserving integro-differential generalization of the conventional local-differential calculus called *isodifferential calculus*, first presented in a systematic way with applications in monographs [61], although implicit in preceding works (e.g., [58]).

Consider a set of functions $f(x), g(x), ...$, on an N-dimensional space $S(x, \mathcal{R})$ with local chart $x = \{x^k\}, k = 1, 2, ..., N$, over the reals $\mathcal{R}(n, +, x)$. Let dx^k and $\partial_k = \partial/\partial x^k$ be the conventional differential and derivative on S, respectively.

Consider now a set of functions $f(x), g(x), ...$, this time, on an N-dimensional isospace $\hat{S}(x, \mathcal{R})$ with the same local chart x but defined over the isofield $\hat{S}(x, \mathcal{R})$ with N-dimensional Class I isounit $\hat{I} = \hat{I}^\dagger = \{\hat{I}_i\} = \{\hat{I}^i_j\} = T^{-1} = [(T^j_i)]^{-1}$ possessing a generally nonlinear-integral dependence on local quantities and their derivatives, $\hat{I}(x, \dot{x}, \ddot{x}, ...)$. *Santilli's isodifferential calculus* is characterized by the *isodifferential*

$$\hat{d}\, x^k = \hat{I}^k{}_i\, dx^i, \qquad ... (2.9)$$

with corresponding *isoderivative*

$$\hat{\partial}_k = \frac{\hat{\partial}}{\hat{\partial} x^k} = T^i_k \frac{\partial_i}{\partial x^k} = T^i_k\, \partial_i, \qquad ... (2.10)$$

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under the condtion that all conventional operations and properties of the ordinary differential calculus are lifted into their isotopic form, e.g.,

$$\hat{d}f(x) = \hat{\partial}_k f * \hat{d}x^k = \partial_i f\, T^i_j\, dx^j, \quad \hat{d}^{\hat{2}}x = \hat{d} * \hat{d}x = \hat{I}\, d^2x,$$

and so on.

A hidden condtion is that, starting with a set of functions over a field $\mathcal{R}(n+,\times)$ with unit $+1$, the operations of isodifferentiation and isoderivatives must preserve the original unit for consistency. This condition remains generally unidentified in the conventional calculus because the preservation of the unit follows from its constancy, $\partial_k I \equiv 0$. For the case of a generalized unit with the same functional dependence of the functions, the conditon of preservation of the unit must be added to the calculus to prevent the transition from the original set of functions defined with respect to $\hat{I}$ to a new set of functions defined over a new unit $\hat{I}'$.

As an example, the defintion of the isodifferential

$$\hat{d}^k_x = d(T^k{}_i\, x^i) = (d\hat{I}^k{}_i)\, x^i + \hat{I}^k{}_i\, dx^i = \hat{I}'^k{}_i\, dx^i, \hat{I}'^k{}_i = (\partial_t \hat{I}^k{}_m)\, x^m + \hat{I}^k{}_i$$

would imply the alteration of the isounit $\hat{I}$. In turn, the occurrence would have serious drawbacks in applications, such as lack of invariance of perturbative series.

Santilli's isodifferential calculus does verify the condition of preserving the basic isounit, although the question whether realizations (2.9) and (2.10) are unique has not been explored until now. Note also the mutual compatibility of isoforms (2.9) and (2.10).

The lifting of the integral calculus follows quite simply from the above isodifferential forms. We here limit ourselves to indicate that an *indefinite isointegral* defined as the operation inverse of the isodifferential is given by

$$\hat{\int} \hat{d}x = \int T\, \hat{I} dx = \int dx = x, \qquad \text{i.e.} \qquad \hat{\int} = \int T. \qquad \ldots (2.11)$$

Note that the isodifferential calculus is one of the simplest possible forms of *integro-differential calculus*, in the sense that each operation has a differential contributions characterized by d or ∂ and an integral component characterized by T or $\hat{I}$, respectively.

Despite its simplicity, the isodifferential calculus has far reaching mathematical and physical implications. Mathematically, it permits a step-by-step generalization of conventional local- differential geometries into covering integro-differential geometries. Physically, the isocalculus permits a generalization of classical and quantum mechanics as well as of their interconnecting map (quantization), as outlined below.

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The *isodual isodifferential calculus* is the antiautomorphic image of the preceding one characterized by the isodual isotopic element $T^d = -T$ or isodual isounit $\hat{I}^d = -\hat{I}$.

The *genodifferential calculus* [61] occurs when the Hermiticity conditon on the isounit is relaxed, $\hat{I} \neq \hat{I}^{\hat{U}}$. As such, the operation of differentiation itself acquires a structural ordering, namely, we have two different genoderivatives $\hat{\partial}^{>} f(x)$ and $f(x)^{<} \hat{\partial}$ defined for the corresponding units $\hat{I}$ and $\hat{I}^{\hat{U}}$ which are naturally set to represent the "arrow of time". This indicates that the genodifferential calculus is significant to represent *irreversible processes*.

2.D. Isospaces, isogeometries and their isoduals. Santilli's third important discovery presented for the first time in paper [51] of 1983 is the isotopic lifting of conventional, N- dimensional, metric (or pseudometric) spaces and related geometries. Consider a metric space $S(x, g, \mathcal{R})$ with local coordinates x and (nowhere singular, real valued) metric g over the reals $\mathcal{R}(n, +, \times)$. Its infinite class of isotopic images over the isoreals, called isospaces (hereon assumed for simplicity to be of Class I) is given by

$$\hat{S}(x, \hat{g}, \hat{\mathcal{R}}) : \hat{g} = Tg, \ \hat{I} = T^{-1}, x^{\hat{2}} = (x^t \hat{g}\, x)\, \hat{I} \in \hat{\mathcal{R}}(\hat{n}, +, \hat{*}), \qquad .. (2.12)$$

where $\hat{g} = g$ is called the *isometric*. The above liftings are necessary for compatibility with the isotopies of the unit $I \to \hat{I}$, of the product $x \to *$ and of the field $\mathcal{R} \to \hat{\mathcal{R}}$ The *isodual isospaces* [51] are given by

$$\hat{S}^d{}_{II}(x, \hat{g}^d, \hat{\mathcal{R}}^d): \hat{g}^d = T^d \eta, \quad T^d = -T, \ x^{\hat{2}d} = (x^t \hat{g}^d\, x)\hat{I}^d \in \hat{\mathcal{R}}^d(\hat{n}^d, +, *^d), \qquad (2.13)$$

where the subindices I or II will be often dropped hereon for simplicity. Note that $x^{\hat{2}} \equiv x^{\hat{2}d}$ and this may be the reason why isodual isospaces remained un-noticed until 1983 [51]. Isospaces and their isodual are however inequivalent, e.g., because the norm of the former is positive-definite, while that of the latter is negative-definite. By recalling that antimatter has been indentified in the negative-definite solutions of Dirac's equation and was originally though to evolve backward in time, isodual isospaces are ideally set to represent antimatter.

Despite their simplicity, isospaces have far reaching implications. In fact, they imply that *the same abstract axioms of conventional spaces (such as the Euclidean, Minskowskian or Riemannian spaces) admit unrestricted functional dependence of the metric*. As an illustration, the conventional Riemannian metric $g(x) \in R(x, g, \mathcal{R})$ is believed to be restricted to the sole dependence on the local coordinates x. Santilli has shown that the same Riemannian axioms permit an unrestricted functional dependence of the metric $\hat{g}(x, \dot{x}, \ddot{x}, ...)$. While Riemanian spaces $R(x, g, \mathcal{R})$ are ideally suited for *exterior* gravitational problems, the *Riemann-Santilli isospaces* $\hat{R}(x, \hat{g}, \mathcal{R})$ are ideally suited for the treatment of interior gravitational

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problems with a nonlinearity in the velocities, integral structure and nonselfadjoint character (Sect. 1).

This remarkable result is due to the construction of the isospaces via the deformation of the metric $g \to \hat{g} = Tg$ while jointly lifting the original unit in the *inverse* amount, $I \to \hat{I} = T^{-1}$, under which isospaces $\hat{S}(x, \hat{g}, \hat{\mathcal{R}})$ (*isodual isospaces* $\hat{S}^d(x, \hat{g}^d, \hat{\mathcal{R}}^d)$) *are locally isomorphic to the original spaces* $S(x, g, R)$, $(S^d(x, g^d, \mathcal{R}^d))$.

Additional salient properties of isospaces are the *preservation of the original dimensionality* and the *preservation of the original basis (except for renormalization factors)* [61a].

Via the use of the isotopies of fields, differential calculus and vector spaces, Santilli's has constructed step-by-step, nonlocal- integral isotopies and isodualities of conventional geometries on metric (or pseudo-metric)spaces. Their most salient application is the geometrization of interior physical media, that is, the geometrization of the departures from empty space caused by matter.

Santilli's isogeometries most important for physical applications are (see[61a] for details): A) The *isoeuclidean geometry* of Class I on three-dimensional isospaces $\hat{E}(x, \hat{\delta}, \hat{\mathcal{R}}), \hat{\delta} = T\delta$, δ = diag. (1, 1, 1), over the isoreals $\hat{\mathcal{R}}(\hat{n}, +, *)$ with isounit which can always be diagonalized into the form

$$\hat{I} = \text{diag.}\ (b_1^{-2}, b_2^{-2}, b_3^{-2}) > 0,\ b_k = b_k(x, \dot{x}, \ddot{x}, \ldots) > 0,\ k = 1, 2, 3.$$

In this case the isometric $\hat{\delta}$ has an arbitrary functional dependence on local coordinates and their derivatives $\hat{\delta}(x, \dot{x}, \ddot{x}, \ldots)$,. Yet the geometry is *isoflat*, that is, it verifies the axioms of flatness in isospace while its projection in the original space $E(x, \delta, \mathcal{R})$ is evidently curved. An intriguing novel notion of the isoeuclidean geometry is the *isosphere of Cass I* $x^{\hat{2}} = (x^t \hat{\delta} x)\hat{I} \in \hat{\mathcal{R}}$ which is perfect sphere in isoeuclidean space. Nevertheless, its projection in the original euclidean space is given by all infinitely possible ellipsoids $xb_1^2x + y_2b^2y + zb_3^2z =$ const. In fact, in isospace the original sphere with radius I is subjected to the deformations of its axes $I_k \to b_k^2$ while the corresponding units are deformed in the *inverse* amounts, $I_k \to b_k^{-2}$, thus preserving the perfectly spherical character. The *isosphere* of Class *III* unifies all compact and noncompact curves $\pm xb_1^2x \pm y_2b^2y \pm zb_3^2z \neq 0$ in isospace. The *isosphere of Class IV* uniflies all compact and noncompact surfaces plus all cones $\pm xb_1^2x \pm y_2b^2y \pm zb_3^2z = 0$. The *isosphere of Class V* is an additional novel notion of a sphere with arbitrary unit (e.g., a lattice).

B) *The isominkowskian geometry of Class I* on isospace $\hat{M}(x, \hat{\eta}, \hat{\mathcal{R}}), \hat{\eta} = T\eta$ over the isoreals with isounit

$$\hat{I} = \text{diag.}\ (b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^2) > 0,\ b_\mu(x, \dot{x}, \ddot{x}, \ldots) > 0,\ \mu = 1, 2, 3, 4.$$

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... (2.15)

which represents *locally varying speeds of light* $c = c_0 b_4 = c_0/n_4$ where c_0 is the speed of light in vacuum and n_4 is the local index of refraction. As such, the isominkowskian geometry is particularly suited for the representation of light propagating wihin inhomogeneous and anisotropic physical media such as our atmosphere. An important notion of the isominkowskian geometry is the *isolight cone of Class I*, which is a perfect cone in isominkowski space but, when projected in the conventional Minkowski space, represents all infinitely possible deformed light cones $xn_1^{-2}x + y_2 n^{-2}y + zn_3^{-2}z - tc_0 n_4^{-2}t = 0$. In fact, each axis of the original ligiht cone is deformed $I_\mu \to n_\mu^{-2}$, while the corresponding units are deformed of the inverse amount, $I_\mu \to n_\mu^2$, thus preserving the original perfect cone. The axiom-preserving character of the isotopies is such that the maximal causal speed of the Minkowski and isominkowski spaces coincide and are given by the speed of light in vacuum c_0.

C) The *isoriemannian geometry* on isospaces $\hat{R}(x, \hat{g}, \hat{R}), \hat{g} = Tg$ over isounit (2.14), which coincides with the conventional geometry at the abstract level. This implies that, unlike the isoeuclidean and isominkowskian geometries, the isoriemannian geometry is *isocurved*, that is curved in isospace. As such, it permits the representation of interior gravitational problesm with locally varying speeds of light such as the bending of light within a physical medium with local speed $c = c_0/n_4 < c_0$, the contribution due to cosmological redshift due to the decrease of the speed of light within astrophysical chromospheres, and others novel insights. An intriguing novel notion is that of isogeodesics of class I which coincide in isospace with the geodesics in vacuum, but when projected in the original Riemannian space represents the actual nongeodesic trajectory within physical media, such as that of a leaf in free fall in our atmosphere.

An isogeometry particularly important for the study of the Lie-Santili theory is the *isosymplectic geometry* first presented in ref. [57]. Consider the conventional symplectric geometry (see, e.g. [34]) in canonical realization on the cotangent bundle $T^*E(r, \delta, T)$, with local chart $a = \{a^\mu\} = \{p_k, r^k\}$, $k = 1, 2, 3$, $\mu = 1, 2, 3, 4, 5, 6$. As well known, the above geometry is characterized by the canonical one form

$$\theta = p_k \, dx^k, \qquad \text{... (2.16)}$$

and the cannocial symplectic two-form

$$w = dp_k \wedge dx^k = \tfrac{1}{2} w_{\mu\nu} \, da^\mu \wedge da^\nu, (w_{\mu\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \text{... (2.17)}$$

which is exact, $w = d\theta$ and therefore closed $dw = d(d\theta) \equiv 0$ (Poincare lemma [34]), with corresponding higher-order forms.

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Santilli's isosymplectic geometry is defined on the *isocotangent bundle* $T*\hat{E}(r, \hat{\delta}, \hat{\mathcal{R}})$ with the same local chart $a = \{p, r\}$ abut now referred to the isounit $\hat{I}_2 = \text{diag.}\{T, \hat{I}\}$, where T appears beacuse referred to the *covariant momentum* p_k. It is characterized by *the one-isoform*

$$\hat{\theta} = p_k \hat{d}x^k = \hat{I}^i_j (x, p, ..) p_i dx^j, \qquad ... (2.18)$$

and the *two-isoform*

$$\hat{w} = \hat{d}p_k \wedge \hat{d}x^k = \tfrac{1}{2} w_{\mu\nu} \hat{d}a^\mu \wedge da^\nu = T^m_k(, p, ...) dp_m \wedge \hat{I}^k_n(x, p, ...) dx^n \equiv w,... \quad (2.19)$$

which is also *isoexact,* $\hat{w} = \hat{d}\hat{\theta}$, and therefore *isoclosed*, $\hat{d}\hat{w} \equiv 0$ (isotopic Poincare' Lemma [57], thus *isosymplectic;* and higher- dimensional isoforms.

Note that $\hat{w} \equiv w$ and this shows the "hidden" character of the isotopies. This also confirms that the *symplectic and isosymplectic geometries coincide at the abstract, realization- free level,* as established by the absract identity of forms (2.15) and (2.18), or (2.17) and (2.19). However, *the two geometries admit inequivalent realizations.* In fact, the symplectic geometry is strictly local-differential, does not admit nonlinearities in the velocities and it is canonical. On the contrary, the isosymplectic geometry has an integro- differential structure (in the sense indicated earlier), it is arbitrarily nonlinear in the velocities and it is noncanonical.

Regrettably, we have to refer the interested reader to monographs [61], [62] and paper [75].

At this writing, the isogeometries are sufficiently well known for physical applications. Nevertheless, their mathematical study is just at the beginning and a number of fundamental aspects remain open for study.

2.E Isotopies and isodualities of functional analysis. As indicated earlier, the isotopies imply nontrivial generalizations of *all* mathematical structures of Lie's theory, inevitably leading to a generalization of functional analysis called by this author *functional isoanalysis* [22].

The generalized discipline begins with the isotopy of continuity (whose knowledge is assumed when dealing with technical aspects of Section 3), and inlcudes the isotopies of conventional square-integrable, Banach and Hilbert spaces, as well as the isotopies of all operations on them.

In particular, functional isoanalysis includes a generalization of conventional special functions, distribbutions and transforms. For instance, the conventional Dirac delta distribution has no meaning under isotopy, mathematically, because of the loss of applicability of the conventional exponentiation and, physically, because particles are no longer point-like. The *isodirace distribution* is the the reconstruction of the conventional distribution for an unrestricted unit

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permitting a direct treatment of the extended character of particles. The Fourier transform, Legendre polynomials, etc., also admit simple yet unique and unambiguous isotopies with important applications in various disciplines.

Regrettably, we are unable to review the above isotopies to prevent a prohibitive length of this paper, and refer the interested reader to [61a]. One should however be aware that the elaboration of the Lie-Santilli isotheory via conventional functional analysis (e.g., the use of conventional trigonometry, logarithms, exponentiations, etc.) leads to inconsistencies which often remain undetected by the noninitiated reader.

2.F: Isotopies and isodualities of classical mechanics. As well known (see, e.g., [13]), Lie's theory admits two fundamental realizations, one in classical and one in quantum mechanics, with interconnecting map given by naive or symplectic quantization.

The preceding isotopies of fields, differential calculus, metric spaces, geometries, and functional analysis were used by Santilli for the construction of step-by step isotopic generalization of classical [62] and quantum [61] mechanics and their interconnecting maps. The new mechanics have been specifically conceived for the most general possible, nonlinear, nonlocal and noncanonical, interior dynamical problems, as well as for the fundamental classical and operator realizations of the Lie- Santilli isotheory. As a matter of fact, Santilli proposed the isotopies of Lie's theory precisely for quantitative treatment of the above generalized mechanics.

It is important to review at least the essential structural elements of the isotopic classical and operator mechanics because they provide the most important realizations of the Lie-Santilli isotheory.

To begin their outline, the isodifferential calculus requires the lifting of both the *space and time isodifferentials*

$$\hat{d}x^k = \hat{I}^k_{s\,i}\, dx^i, \quad \hat{d}t = \hat{I}_t\, dt, \quad \hat{d}x_k = T^i_k\, dx_i,, \quad \hat{d}p_k = T^i_k\, dp_k, \quad \text{... (2.20)}$$

where $\hat{s^I} = ({}_sT)^{-1} = \hat{I}_t = ({}_tT)^{-1}$ are the *space and time isounits with corresponding space and time isotopic elements* ${}_sT$ and $\hat{I}_t = T_t^{-1}$. The corresponding *space and time isoderivatives* are given by

$$\partial_k = \hat{\partial}/\hat{\partial}\, x^k = T^i_k\, \partial_i, \quad \hat{\partial}_t = T_t\, \partial_t, \quad \hat{\partial}^k = \hat{\partial}/\hat{\partial}x_k = \hat{I}^k_i\, \partial^i. \quad \text{... (2.21)}$$

The application of the isodifferential calculus to classical Hamiltonian mechanics then implies the following generalizations :

1) Newton's equations of motion $md^2x_k/dt^2 = -\partial_k V + F^{NSA}_k$ on $E(x, \delta, \mathcal{R})$ over $\mathcal{R}(n, +, \times)$ are lifted into the Newton- Santilli equations on the isoeuclidean space $\hat{E}(x, \hat{\delta}, \hat{\mathcal{R}})$

$$\hat{m}\hat{d}^2\, x_k/\hat{d}t^2 = -\hat{\partial}_k\, V(x), \quad \text{... (2.22)}$$

which, when projected in the original space, read

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$$mT_t^2 \frac{d^2({}_S T_k \hat{\delta}_{ij} x^j)}{dt^2} = - {}_s T^i_k \frac{\partial V(\hat{x})}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x^i}; \qquad \ldots (2.23)$$

2) the conventional canonical action $\int_{t_1}^{t_2} (p_k dx^k - Hdt)$ is lifted into the isoaction

$$\hat{A} = \int_{t_1}^{t_2} [p_k \hat{d} x^k - H \hat{d} t] = \int_{t_1}^{t_2} dt\, L(t, x, \dot{x}, \dot{x}, \ldots) =$$

$$= \int_{t_1}^{t_2} dt\, [p_{ks} \hat{I}^k_i (t, x, p, \ldots) dr^i - H_t T(t, x, p, \ldots) dt]; \qquad \ldots (2.24)$$

3) Hamilton's equations without external terms are lifted into the *Hamilton-Santilli* equations

$$\frac{\hat{d} x^k}{\hat{d} t} = \frac{\hat{\partial} \hat{H}(t, x, p)}{\hat{\partial} p_k}, \quad \frac{\hat{d} p_k}{\hat{d} t} = \frac{\hat{\partial} \hat{H}(t_1 x_1 p)}{\hat{\partial} x^k}, \qquad \ldots (2.25)$$

4) The conventional Lagrange equations and underlying Legendre transform are lifted into the form

$$\frac{\hat{d}}{\hat{d} t} \frac{\hat{\partial} \hat{L}(t, x, \dot{x})}{\hat{\partial} \dot{x}^k} - \frac{\hat{\partial} \hat{L}(t, x, \dot{x})}{\hat{\partial} x^k} = 0, \; p_k = \frac{\hat{\partial} \hat{L}}{\partial \dot{x}^k}, \; \dot{x} = \hat{d} x / \hat{d} t, \qquad \ldots (2.26)$$

5) The Hamilton-Jacobi equations are lifted into the *isotopic Hamilton-Jacobi equaitons*

$$\hat{\delta}_t \hat{A} + \hat{H} = 0, \quad \hat{\partial} / \hat{\partial} r^k - p_k = 0, \; \hat{\partial} \hat{A} / \hat{\partial} p_k \equiv 0; \qquad \ldots (2.27)$$

6) The conventional Poisson brackets, which are the realization in classical mechanics (CM) of the Lie product, are lifted into the *isopoisson* brackets first introduced in ref. [47] (see also [57] for the explicit form below)

$$[\hat{A}, \hat{B}]_{CM} = \frac{\hat{\partial} \hat{A}}{\partial x^i} {}_s T^i_j \frac{\hat{\partial} \hat{B}}{\hat{\partial} p_j} - \frac{\hat{\partial} \hat{B}}{\hat{\partial} x^i} T^i_{s\,j} \frac{\hat{\partial} \hat{A}}{\hat{\partial} p_j} = \frac{\partial A}{\partial x^i} {}_s T^i_j \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial x^i} {}_s T^i_j \frac{\partial B}{\partial p_j}, \qquad \ldots (2.28)$$

which provide the desired classical realization of the Lie- Santilli brackets. In fact, it is easy to prove that the above brackets satisfy in isospace (but not in the original space) the Lie algebra axioms (see ref.s [61], [62] for a proof via the isotopies of the Poincare′ lemma in the symplectic geometry).

(7) The classical realization of the exponentiated form of Hamilton's equation as a realization of a one parameter Lie transformation group is lifted into the exponentiated from of Eq.s (2.25)

$$a' = \{\hat{e^{tw^{\mu\nu}}} \hat{\partial}_\mu \hat{H} \hat{\partial}_\nu\} * \times = e^{tw^{\mu\nu} T^\alpha_\nu \partial_\mu H \partial_\alpha} a, \; a = \{p, r\}, \qquad \ldots (2.29)$$

which is a realization of the one-dimensional Lie-isotopic groups, as we shall see in the next section.

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The emerging new mechanics is called *Hamilton-Santilli* isomechanics. Some of the advantages over the conventional Hamiltonian mechanics are the following :

1) In Newton's equations of motion the bodies are approximated as massive points, as well known since Newton's time, and the same approximation persists at in Hamiltonian mechanics. The Newton-Santilli equations can represent instead the actual nonspherical shape of the body considered as well as its possible deformations. As a simple illustration one can assume for isotopic element the matrix with constant elements ${}_sT = \text{diag.}\ (a^{-2}, b^{-2}, c^{-2}, a, b, c \neq 0,$ which represents in isoeuclidean space a spheroidal ellipsoid. More complex shapes can be represented with nondiagonal isotopic elements. Deformations of the original shape due to external fields can be represented with nondiagonal isotopic elements. Deformations of the original shape due to external fields can be represented via suitable dependence of the isotopic element. The same representation of shape evidently persists in the Hamilton-Santilli equations (2.25) [see [61] for details]).

2) Only some of Newton's equations of motion are representable with Hamilton's equations in the cordinates x of the observer because they are in general *essentially noneselfadjoint* [49b]. In this case a Hamiltonian representation requires certain coordinates transformations $x \to x'(t, x, p), p \to p'(t, x, p)$ which are nonlinear, and therefore not realizable in actual experiment. The Hamilton- Santilli equations do permit instead the representation of these systems in the actual coordinates x of the observer. In fact, Eq.s (2.25) are variationally selfadjoint in isospace but not in regular space. A such, Eq.s (2.25) achieve an analytic representation in isoeuclidean space of systems which do not admit one in the original Euclidean space (see also [61] for details).

3) Newton's equations for bodies moving within a resitive medium are integro-differential

$$m\ddot{x} = -\partial V + \int_{\sigma} d\sigma \ \mathcal{F}(\sigma, ...), \qquad ... (2.30)$$

namely, they are differential with respect to the trajectory x of the centre of mass, while admitting corrections due to the shape σ of the body where $\mathcal{F}$ is a suitable kernel. The representation of these systems via Hamilton's equations is prohibited because the toplogy of the underlying symplectic geometry is strictly local-differential. The Hamilton-Santilli equations (2.25) can instead represent the above integro- differential systems because the underlying isosymplectic geometry does indeed admit such systems.

2.G: Isotopies of quantum mechanics. We now outline the operator realization of the Lie-Santilli isotheory first identified in ref. [48] and then studied in numerous subsequent papers (see ref. [61b] for the most recent account).

The isotopies of quantum mechanics were originally proposed by Santilli [48] under the name of *hadronic mechanics*, namely, a mechanics built for strongly interacting particles called hadrons. Since the charge radius of hadrons is of the same order of magnitude of the range of the strong interactions, and hadrons are the densest objects

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measured in laboratory under now, the activation of the strong interactions requires the mutual penetratrion of these hyperdense particles. This implies the expectation of nonlinear, nonlocal-integral and nonhamiltonian contributions in the strong interactions resulting in the need for a suitable generalization of quantum mechanics.

Let ξ be the enveloping associative operator algebra of quantum mechanics with elements $A, B, ..$ unit I and conventional associative product AB, and let $\mathcal{H}$ be the conventional Hilbert space with states $|\psi>, |\phi>, ...$ and inner product $<\psi \mid \phi> = \int d^3 r \psi^{\tau}(t, r)\phi(t, r)$ over the complex numbers $C(c, +, \times)$.

Hadronic mechanics is based on the lifting of the (sapce) unit $I \to \hat{I} = T^{-1}$ with consequential lifting of ξ into the *enveloping isoassociative algebra* $\hat{\xi}$ with the same elements $A, B, ...$ now equipped with the isoassociative product $A * B = ATB$, and lifting of $\mathcal{H}$ into the *isohilbert space* $\hat{\mathcal{H}}$ with *isostates* $|\hat{\psi}>, |\hat{\phi}>, ...$ and *isoinner product* over the isofield of complex numbers

$$\hat{\mathcal{H}} : <\hat{\phi} \uparrow \hat{\psi}> = <\hat{\phi} \mid T \mid \hat{\psi}> \hat{I} \in \hat{C}(\hat{c}, +, *). \qquad ... (2.31)$$

The fundamental dynamical equations of hadronic mechanics can be uniquely and unambigously derived from the Hamilton-Santilli isomechanics via the isotopies of conventional or symplectic quantization. Recall that the *naive quantization* can be expressed via the mapping of the canonical action

$$A = \int_*^{t_2} (p dx^k - H\, dt) \to -ih\, Ln\psi(t, r). \qquad ... (2.32)$$

Such a mapping is now inapplicable to isoaction (2.24) because $\hat{A} \neq A$. But $h = 1$ is the basic unit of quantum mechanics which is replaced under isotopies by the (space) isounit $\hat{I}$. The consistent application of the isotopies then yields the generalized mapping identified by Animalu and Santilli and here presented for simplicity for the isounit independent from the local time and coordinates (but dependent on the velocities as essential for contact resistive forces, see [61b] for the general case and references)

$$\hat{A} = \int_{t_1}^{t_2} [p_k \hat{d} r^k - H \hat{d} t] \to -i\hat{I}\, Ln\, \psi\, (t, r), \qquad ... (2.33)$$

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characterizing the *naive isoquantization* of generalized Hamilton-Jacobi equations (2.27) into the following fundamental dynamical equations of hadronic mechanics (see ref. (61b) for all references and details): *the isoschrödinger equations for the linear momentum*

$$-i\hat{\partial}_k \psi(t,r) = -iT^i_k\, \partial_i\, \psi(t,r) = p_k * \psi(t,r) = p_k\, T\psi(t,r), \qquad \ldots (2.34)$$

with the related *fundamental isocommutation rules*

$$p_u * r^j - r^j * p_i = -\hat{\delta}^j_i = -\hat{I}^j_i; \qquad \ldots (2.35)$$

first identified by Santilli; the *isoscrödinger equation for the energy*

$$i\hat{\partial}_t\, \psi\,(t,r) = i_t T\, \partial t_t\, \hat{\psi} pt, r) = \hat{H} * \psi(t,r) = \hat{H}\, T_s \psi(t,r) = \hat{E} * \psi(t,r) = E\psi(t,r), \qquad \ldots (2.36)$$

first identified by Myung and Santilli with the conventional differential calculus, and finalized by Santilli with the isoderivatives; and the *isoheisenberg equation*

$$i\hat{d}Q/\hat{d}t = [Q,\hat{H}] = Q * H - H * Q = QTH - HTQ. \qquad (2.37)$$

with integrated form

$$Q(t) = e^{iHTt}\, Q(0) e^{-itTH}, \qquad \ldots (2.38)$$

first identified by Santilli in the original proposal to build hadronic mechanics [48].

It should be recalled for subsequent need that *the condition of isohermiticity on an isohilbert space coincide with the conventional Hermiticity* $\hat{X^\tau} = X^\tau$. As a *consequence, all opeartors which are Hermitean in quantum mechanics remain so in hadronic mechanics.*

Also, unitary transforms on $\mathcal{H}$, $U\,U^\tau = U^\tau = I$,are lifted under isotopies into the *isounitary transformstions*

$$\hat{U} * \hat{U}^\tau = \hat{U}^\tau * U = \hat{I}.$$

The *fundamental operator realization of the Lie-Santilli isoproduct* is therefore givenby $[A, B] = ATB - BTA$ where A, B are elements of the isoenvelope $\hat{\xi}$ on an isohilbert space $\hat{\mathcal{H}}$ over the isofield $\hat{C}(\hat{c}, +, *)$. The *fundamental operator realization of the isogroups* is then given by Eq. (2.38) which, as we shall see, can be rewritten in terms of the isounitary transforms.

The advantages of hadronic over quantum mechanics are similar to those of the Hamilton-Santilli over the Hamiltonian mechanics. In fact, quantum mechanics is local-differential-potential and, as such, can only represent (in first quantization) point particles under action-at-a-distance interactions. By comparison, hadronic mechanics can represent (in first isoquantization) the actual nonspherical shape of the particles, all their infinitely possible deformations as well as nonlocal-integral interactions due to mutual penetration of the

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wavepackets of hadrons. The possibilities for broader applications in various disciplines is then evident.

2.E: Isolinearity, isolocality and isocanonicity. In Sect. 1 we pointed out that the primary limitations of the contemporary formulation of Lie's theory for applications are those of being linear, local and canonical. The classical realizations identified earlier indicate rather clearly that the Lie-Santilli isotheory is nonlinear, nonlocal and noncanonical, as desired.

It is important to understand that such nonlinearity, nonlocality and noncanonicity occur only when the theory is projected in the original space over the original fields because the theory reconstructs linearity, locality and canonicity in isospace (see [61] for all details and references).

Let $S(x, F)$, be a conventional vector space with local coordinates x over a field F, and let $x' = A(w)x$ be a linear, local and canonical transformation on $S(x, F)$, $w \in F$. The lifting $S(x, F) \to \hat{S}(x, \hat{F})$ requires a corresponding necessary isotopy [47]

$$x' = \hat{A}(\hat{w}) * \hat{A}(\hat{w})Tx, \; T \text{ fixed}, \; x \in \hat{S}(x, \hat{F}), \; \hat{w} \quad I = T^{-1}, \qquad \ldots (2.39)$$

called *isotransforms*, with isodual $x' = \hat{A}^d(\hat{w}^d)*^d{}_x = -\hat{A}(\hat{w})*x$.

It is easy to see that isotransforms satisfy the condition of linearity in isospaces, called by Santilli *isolinearity*

$$A * (\hat{a} * \times + \hat{b} * y) = \hat{a} * (A * x) + \hat{b} * (A * y), \; \forall x, y \in \hat{S}(x, \hat{F}), \quad \hat{a}, \hat{b} \in \hat{F}. \qquad \ldots (2.40)$$

althouth their projection in the original space $S(x, F)$ the isotransforms are nonlinear, because $x' = \hat{A}T(x, \dot{x}, \ldots)x$. Isotransforms (2.39) are also *isolocal* because the theory formally deals with the local variables x while all nonlocal terms are embedded in the isounit, namely, all nonlocal-integral terms disappear at the abstract, realization-free level. Nevertheless, the theory is nonlocal when projected in the original space. Similarly, isotopic theories are *isocanonical* because they are from the isoaction (2.24) which coincides at the abstract level with the canonical action.

3. Isotopies and Isodualities of Enveloping Algebras, Lie Algebras, Lie Groups, Symmetries, Representation Theory and Their Applications

As recalled in Sect.1, Lie's theory (see, e.g.,[13], [15] and [76]) is centrally dependent on the basic n-dimensional unit I= diag. (1, 1 ,..., 1) in all its major branches, such as enveloping algebras, Lie algebras, Lie groups, representation theory, etc. The main idea of the Lie-santilli theory [47], [49], [61], [62] is the reformulation of the entire conventional theory with respect to the most general possible, integro-differential isounit $\hat{I}(x, \dot{x}, \ddot{x}, \ldots)$.

One can therefore see from the very outset the richness and novelty of the isotopic theory. In fact, it can be classified into five main

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classes as occurring for isofields, isospaces, etc., and admits novel realizations and applications, e.g., in the construction of the symmetries of deformed line elements of metric spaces.

3.A. Isotopies and isodualities of universal enveloping associative algebras. Let ξ be a universal enveloping associative algebra [15] over a field F (of characteristic zero) with generic elements $A, B, C, ...$, trivial associative product AB and unit I. Their isotopes $\hat{\xi}$ were first introduced in [47] under the name of *isoassociative envlopes*. They coincide with ξ as vector spaces but are equipped with the isoproduct so as to admit $\hat{I}$ as the correct (right and left) unit

$$\hat{\xi}: A * B = ATB, \ T \text{ fixed}, I * A = A * I \equiv A \ \forall A \in \hat{\xi}, \hat{I} = T^{-1}. \quad ... (3.1)$$

Let $\xi = \xi(L)$ be the universal enveloping algebra of an N-dimensional Lie algebra L with ordered basis $\{X_k\}, k = 1, 2, ..., N$, $[\xi(L)]^- \approx L$ over F, and let the infinite-dimensional basis of $\xi(L)$ be given by the Poincare- Birkhoff-Witt theorem [15]. A fundamental result achieved by Santilli in the original proposal [47] (see also [59, Vol. II,p. 154-163] is the following

Theorem 3.1. *The cosets of $\hat{I}$ and the standard, isotopically mapped monomials*

$$\hat{I}, X_k \ X_i * X_j \ (i \le j), \ X * X_j * X_k \ (i \le j \le k), ... \quad ... (3.2)$$

form a basis of the universal enveloping isoassociative algebra $\hat{\xi}(L)$ of a Lie algebra L.

A first important consequence is that the isotopies of conventional exponentiation are given by the expression, called isoexponentiation, for $\hat{w} \in \hat{F}$,

$$e^{i\hat{w}*X}_{\hat{\xi}} = \hat{I} + (i\hat{w}*X)/(1! + (i\hat{w}*X) * (i\hat{w}*X)/2! + ... = \hat{I}\{e^{iwTX}\} = \{e^{iXTw}\}\hat{I}. \quad ... (3.3)$$

As anticipated in Section1, the nontriviality is illustrated by the emergence of the nonlinear-nonlocal isotopic element T directly in the exponent of the transformations, thus ensuring the desired generalization.

The implications of Theorem 3.1 also emerge at the level of functional analysis because all structures defined via the conventional exponentiation must be suitably lifted into a form compatible with Theorem 3.1. As an example, Fourier transforms are structually dependent on the conventional exponentiation. As a result, they must be lifted under isotopies into the expression [23]

$$f(x) = (1/\pi) \int_{-\infty}^{+\infty} g(k) * e^{ikx}_{\xi} \, dk, g(k) = (1/2\pi) \int_{-\infty}^{+\infty} f(x) * e^{-ikx}_{\xi} \, dx, \quad ... (3.4)$$

with similar liftings for Laplace transforms, Dirac-delta distribution, etc., not reviewed here for brevity.

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On physical grounds, Theorem 3.1 implies that the isotransform of a Gaussian in isofunctional analysis is given by [23]

$$f(x) = n * e_{\xi}^{-x/2a^2} = Ne^{-x^2T/2a^2} \Rightarrow g(k) = N * e\xi^{-k^2a^2/2} = N'e^{-k^2Ta^2/2}. \quad \ldots (3.5)$$

As a result, the widths are of the type $\Delta x \approx aT^{-1/2}$, $\Delta k \approx a^{-1}T^{-1/2}$. It then follows that the isotopies imply the loss of the conventional uncertainties $\Delta x\, \Delta k \approx 1$ in favour of the local *isouncertainties* [61b].

$$\Delta x\, \Delta k \approx \hat{I}, \quad \ldots (3.6)$$

which illustrate the nontriviality of the isotopy.

The *isodual isoenvelopes* $\hat{\xi}^d$ are characterized by the isodual basis $X_k^d = -X_k$ defined with respect to the isodual isounits $\hat{I}^d = -\hat{I}$ and isodual isotopic element $T^d = -T$ over the isodual isofields $\hat{F}^d$. The *isodual isoex-ponentiation* is the given by.

$$e_{\hat{\xi}^d}^{i^d w^d x\, dX^d} = \hat{I}^d \{e^{iwTX}\} = -e_{\hat{\xi}}^{iwX} \quad \ldots (3.7)$$

and plays an important role for the characterization of antiparticles as possessing negative-definite energy and moving backward in time(as necessary when using isodual isofields).

It is easy to see that Theorem 3.1 holds, as originally formulated [47], for envelopes now called of Class III, thus unifying isoenvelopes $\hat{\xi}$ and their isoduals $\hat{\xi}^d$. In fact, the theorem was conceived to unify with one single Lie algebra basis X_k nonisomorphic compact and noncompact algebras of the same dimension N (see the example of Section 3.E).

The isotopy $\xi \to \hat{\xi}$ is not a conventional map because the local coordinates x, the infinitesimal generators X_k and the parameters w_k are not changed by assumption, while the underlying unit and related associative product are changed. Also, in the operator realization the Lie and Lie-Santilli isotheory can be linked by nonunitary transformations $UU^\tau = \hat{I} \neq I$, for which.

$$I \to \hat{I} = UIU^\tau, AB - UABU^\tau = A' * B' = A'\, TB', T = (UU^\tau)^{-1} \quad \ldots (3.8)$$

where $A' = UAU^\tau, B' = UBU^\tau$. The lack of equivalence of the two theories is further illustrated by the inequivalence between conventional eigenvalue equations.

$$H \mid b> = E \mid b>,\ H = H^\tau,\ E \in \mathcal{R}(n, +, \times),$$

and their isotopic form in the same Hamiltonian

$$H * \mid \hat{b}> = HT \mid \hat{b}> \equiv \hat{E} \mid \hat{b}>,\ H = H^\tau,\ E' \in \mathcal{R}(n, +, \times),$$

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with consequential *different eigenvalues for the same operator* $H, E' \neq E$ (see Section 3.E for an example). We therefore expect the weights of the Lie and Li-Santilli theories to be different.

3.B. Isotopies and isodualities of Lie algebras. A (finite-dimensional) isospace $\hat{L}$ over the isofield $\hat{F}$ of isoreal $\hat{\mathcal{R}}(\hat{n}, +, *)$ or isocomplex numbers with isotopic element T and isounit $\hat{I} = T^{-1}$ is called a *Lie -Santilli algebra* over $\hat{F}$ (see [47], [49], [61], [62] for original studies and monographs [3], [24], [31], [74] with large number of quoted papers for independent studies), sametimes called isoalgebra (when no confusion with the isotopies of non- Lie algebras arises), when there is a composition $[A, \hat{,} B]$ in $\hat{L}$, called *isocommutator*, which is isolinear (i.e. satisfies condition (2.40)) and such that for all $A, B, C \in \hat{L}$

$$[A, \hat{,} B] = -[B, \hat{,} A], \quad [A, \hat{,} [B, \hat{,} C]] + [B, \hat{,} [C, \hat{,} A]] + [C, \hat{,} [A, \hat{,} B]] = 0, \qquad \ldots (3.9a)$$

$$[A * B, \hat{,} C] = A * [B, \hat{,} C] + [A, \hat{,} C] * B. \qquad \ldots (3.9b)$$

The isoalgebras are said to be *isoreal (isocomplex)* when $\hat{F} = \hat{\mathcal{R}}(\hat{F} = \hat{C})$, and *isoabelian* when $[A, \hat{,} B] \equiv 0 \; \forall \; A, B \in \hat{L}$. A subset $\hat{L}_0$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when $[\hat{L}_0, \hat{,} \hat{L}_0] \subset \hat{L}_0$ and an isoideal when $[\hat{L}, \hat{,} \hat{L}_0] \subset \hat{L}_0$. A maximal isoideal which verifies the property $[\hat{L}, \hat{,} \hat{L}_0] = 0$ is called the *isocentre* of $\hat{L}$. For the isotopies of conventional notions, theorems and properties of Lie algebras, one may see monograph [74].

We recall the *isotopic generalizations of the celebrated Lie's First, Second and Third Theorems* introduced in the original proposal [47], but which we do not review here for brevity (see [49b], [61b], [74]. For instance, the isotopic second theorem reads

$$[X_i, \hat{,} X_j] = X_i * x_j - X_j * X_i = X_i \, T(x, \ldots) \, X_j - X_j \, T(x, \ldots) \, X_i = \hat{C}_{ij}^{k} \, (x, \dot{x}, \dot{x}, \ldots) * X_k,$$

where the $\hat{C}'s$ are called the *structure functions*, generally have an explicit dependence on the underlying isospace (see the example of Section 3.E.), and verify certain restrictions from the Isotopic Third Theorem.

Let L be an N-dimensional Lie algebra with conventional commutation rules and structure constants C_{ij}^{k} on a space $S(x, F)$ with local coordinates x over a field F, and let L be (homomorphic to) the antisymmetric algebra $[\xi(L)]^-$ attached to the associative envelope $\xi(L)$. Then $\hat{L}$ can be equivalently defined as (homomorphic to) the antisymmetric algebra $[\hat{\xi}(L)]^-$ attached to the isoassociative envelope $\hat{\xi}(L)$ ([47], [49], [74]). In this way, an infinite number of isoalgebras $\hat{L}$, depending on all possible isounits $\hat{I}$, can be constructed via the isotopies of one single Lie algebra L. It is easy to prove the following result:

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Theorem 3.2 *The isotopies $L \to \hat{L}$ of an N-dimensional Lie algebra L preserve the original dimensionality.*

In fact, the basis $e_k, k = 1, 2, ..., N$ of a Lie algebra L is not changed under isotopy, except for renormalization factors denoted $\hat{e}_k$ Let the commutation rules of L be given by

$$[e_i, e_j] = C_{ij}^k e_k.$$

The isocomutation rules of the isotopes $\hat{L}$ are

$$[\hat{e}_i, \hat{e}_i] = \hat{e}_i T \hat{e}_j - \hat{e}_j T \hat{e}_i = \tilde{C}_{ij}^k (x, \dot{x}, \dot{x}, ...) \hat{e}_k, C = \hat{C}T. \quad ... (3.11)$$

One can then see in this way the necessity of lifting the structure <constants> into structure<functions> , as correctly predicted by the Isotopic Second Theorem.

The structure theory of the above isoalgebras is still unexplored to a considerable extent. In the following we shall show that the main lines of the conventional structure of Lie theory do indeed admit a consistent isotopic lifting. The begin, we here introduce the *general isolinear and isocomplex Lie-Santilli algebras* denoted $G\hat{L}(n, \hat{C})$ as the vecor isospaces of all $n \times n$ comlex matrices over $\hat{C}$. It is easy to see that they are closed under isocommutators as in the conventional case. The *isocentre* of $G\hat{L}(n, \hat{C})$ is then given by $\hat{a} * \hat{I}, \forall \hat{a} \in \hat{\mathcal{R}}$. The subset of all complex $n \times n$ matrices with null trace is also closed under isocommutators. We shall call it the *special, complex, isolinear isoalgebra* and denote it with $S\hat{L}(n, \hat{C})$. The subset of all antisymmetric $n \times n$ real matrices $X, X^t = -X$, is also closed under isocommutators, it is called the *isoorthogonal algerba*, and it is denoted with $\hat{O}(n)$.

By proceeding along similar lines, we classify all classical, non-exceptional, Lie-Santilli algebras over an isofield of characteristic zero into the isotopes of the conventional forms, denoted with $\hat{A}_n, \hat{B}_n, \hat{C}_n$ and $\hat{D}_n$ each one admitting realizations of Classes I, II, III, IV and V (of which only Classes I, II and III are studied herein). In fact $\hat{A}_{n-1} = S\hat{L}(n, \hat{C}); \hat{B}_n = \hat{O}(2n + 1, \hat{C}); \hat{C}_n = S\hat{P}(n, \hat{C});$ and $\hat{D}_n = \hat{O}(2n, \hat{C})$.One can begin to see in this way the richness of the isotopic theory as compared to the conventional theory.

The notions of *homomorphism, automorphism and isomorphism* of two isoalgebras $\hat{L}$ and $\hat{L}'$, as well as of *simplicity and semisimplicity* are the conventional ones. Similarly, all properties of Lie algebras based on the addition, such as the *direct and semidirect sums, carry* over to the isotopic context unchanged (because of the preservation of the conventional additive unit 0).

An *isoderivation* $\hat{D}$ of an isoalgebra $\hat{L}$ is an isolinear mapping of $\hat{L}$ into itself satisfying the property

$$\hat{D}([A, B]) = [\hat{D}(A), B] + [A, \hat{D}(B)] \forall A, B \in \hat{L}. \quad ... (3.12)$$

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If two maps $\hat{D}_1$ and $\hat{D}_2$ are isoderivations, then $\hat{a} * \hat{D}_1 + \hat{b} * \hat{D}_2$ is also an isoderivation, and the isocommutators of $\hat{D}_1$ and $\hat{D}_2$ is also an isoderivation. Thus, the set of all isoderivations forms a Lie-Santilli algebra as in the conventional case.

The isolinear map $\hat{ad}(\hat{L})$ of $\hat{L}$ into itself defined by

$$\hat{ad}\, A(B) = [A \hat{,} B],\ \forall\, A, B \in \hat{L}, \qquad \text{... (3.13)}$$

is called the *isoadjoint map*. It is an isoderivation, as one can prove via the iso-Jacobi identity. The set of all $\hat{ad}(A)$ is therefore an isolinear isoalgebra, called *isoadjoint algebra* and denoted $\hat{L}_a$. It also results to be an isoideal of the algebra of all isoderivations as in the conventional case.

Let $\hat{L}^{(0)} = \hat{L}$. Then $\hat{L}^{(1)} = [\hat{L}^{(0)} \hat{,} \hat{L}^{(0)}]$, $\hat{L}^{(2)} = [\hat{L}^{(1)}, \hat{L}^{(1)}]$, etc., are also isoideals of $\hat{L}.\hat{L}$ is then called *isosolvable* if, for some positive integer n, $\hat{L}^{(n)} \equiv 0$. Consider also the sequence.

$$L_{(0)} = L,\ \hat{L}_{(1)} = [\hat{L}_{(0)} \hat{,} \hat{L}],\ \hat{L}_{(2)} = [\hat{L}_{(1)} \hat{,} \hat{L}],\ \text{etc.},$$

Then $\hat{L}$ is said to be *isonilpotent* if, for some positive integer $n, \hat{L}(n) \equiv 0$. One can then see that, as in the conventional case, an isonilpotent algebra is also isosolvable, but the converse is not necessarily true.

Let the isotrace of a matrix be given by the element of the isofield [61]

$$\hat{Tr} A = (Tr\, A)\hat{I} \in \hat{F}, \qquad \text{... (3.14)}$$

where $Tr\, A$ is the conventional trace. Then

$$\hat{Tr}\,(A * B) = (\hat{Tr}\, A) * (\hat{Tr}\, B),\ \hat{Tr}\,(BAB^{-1}) = \hat{Tr}\, A.$$

Thus, the $\hat{Tr}\, A$ preserves the axioms of $Tr\, A$, by therefore being a correct isotopy. Then the isoscalar product

$$(A \hat{,} B) = \hat{Tr}[(\hat{Ad} X) * (\hat{ad}\, B)], \qquad \text{... (3.15)}$$

is here called the *isokilling form*. It is easy to see that $A \hat{,} B)$is symmetric, bilinear, and verifies the property $\hat{Ad} X(Y) \hat{,} Z) + (Y \hat{,} Ad\, X(z)) = 0$, thus being a correct, axiom-preserving isotopy of the conventional Killing form.

Let e_k, $k = 1, 2, \ldots N$, N, be the basis of L with oneto-one invertible map $e_k \to \hat{e}_k$ to the basis of $\hat{L}$. Generic elements in $-\hat{L}$ can then be written in terms of local coordinates $x, y, z, A = x^i \hat{e}_i$ and $B = y^j \hat{e}_j$, and

$$C = z^k\, \hat{e}_k = [A \hat{,} B] = x^i\, y^j\, [\hat{e}_i \hat{,} \hat{e}_j] = x^i\, x^j\, \tilde{C}^k_{ij}\, \hat{e}_k.$$

Thus,

$$[\hat{Ad}\, A(B)]^k = [A \hat{,} B]^k = \tilde{C}^k_{ij}\, x^i\, x^j. \qquad \text{... (3.16)}$$

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We now introduce the *isocartan tensor* $\tilde{g}_{ij}$ of an isoalgebra $\hat{L}$ via the definition $(A\,\hat{,}\,B)\,(B) = \tilde{g}_{ij}\,x^i\,y^j$ yielding

$$\tilde{g}_{ij}\,(x, \dot{x}, \ddot{x}, \ldots) = \tilde{C}^k_{ip}\,C_{jk}\,p. \qquad \ldots (3.17)$$

Note that the isocartan tensor has the general dependence of the isometric tensor of Section 2.C, thus confirming the inner consistency among the various branches of the isotopic theory. In particular, the isocartan tensor is generally *nonlinear, nonlocal and noncanonical* in all variables $x, \dot{x}, \ddot{x}, \ldots$. This clarifies that isotopic generalization of the Riemannian spaces studied in the companion paer [60] $\hat{R}(\hat{x}, \hat{g}, \hat{R})$, $\hat{g} = \hat{g}(s, x, \dot{x}, \ddot{x}, \ldots)$, has its origin in the very structure of the Lie-isotopic theory.

The isocartan tensor also clarifies another fundamental point of Section 1, that the isotopies naturally lead to an arbitrary dependence in the velocities and accelerations, exactly as needed for realistic treatment of the problems identified in Section 1, and that their restriction to the nonlinear dependence on the coordinates x only, as generally needed for the exterior(e.g., gravitational)problem, would be manifestly un-necessary.

The isotopies of the remaining aspects of the structure theory of Lie algebras can be completed by the interested reader. Here we limit ourselves to recall that when the isocartan form is positive- (or negative-) definite, $\hat{L}$ is compact, otherwise it is noncompact. Then it is easy to prove the following.

Theorem 3.3 *The Class III liftings $\hat{L}$ of a compact (noncompact) Lie algebra L are not necessarily compact (noncompact).*

The identification of the remaining properties which are not preserved under liftings of Class III is an instructive task for the interested reader. For instance, if the original structure is irreducible, its isotopic image is not necessarily so even for Class I, trivially, because the isounit itself can be reducible, thus yielding a reducible isotopic structure.

Let $\hat{L}$ be an isoalgebra with generators X_k and isounit $\hat{I} = T^{-1} > 0$. From Equations (3.7) we then see that the *isodual Lie-Santilli algebras* $\hat{L}^d$ of $\hat{L}^d$ is characterized by the isocommutators

$$[X_i\,\hat{,}\,X^d_{i]} = -\,[X_i\,\hat{,}\,X_j] = \tilde{C}^{k\,(d)}_{ij}\,X^d_k,\ \tilde{C}^{k(d)}_{ij} = -\,\tilde{C}^k_{ij}. \qquad \ldots (318)$$

$\hat{L}$ and $\hat{L}^d$ are then (anti) isomorphic. Note that the isoalgebras of Class III contain all class I isoalgebras $\hat{L}$ and all their isoduals $\hat{L}^d$. The above remarks therefore show that the Lie-Santilli theory can be naturally formulated for Class III, as implicity done in the original proposal [47]. The formulation of the same theory for Class IV or V is however considerably involved on technical grounds thus requiring specific studies.

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The notion of isoduality applies also to conventional Lie algebras L, by permitting the identification of the *isodual Lie algebras* L^d via the rule [52], [53]

$$[X_i, X_j]^d = X^d_i\, I^d X^d_j - X^d_j\, I^d X^d_i = -\,[X_i, X_j] = C^{k(d)}_{ij}\, X^d_k,\ C^{k(d)}_{ij} = -\,C^k_{ij}.$$

Note the necessity of the isotopies for the very construction of the isodual of conventional Lie algebras. In fact, they require the nontrivial lift of the unit $I \Rightarrow I^d = (-I)$, with consequential necessary generalization of the Lie product $AB - BA$ into the isotopic form $ATB - BTA$.

The following property is mathematically trivial, yet carries important physical applications.

Theorem 3.4 *All infinitely possible, Class I isotopes $\hat{L}$ of a (finite- dimensional) Lie algebra L are locally isomorphic to L and all infinitely possible, Class II isodual isotopes $\hat{L}^d$ are anti- isomorphic to L.*

As outlined in Section 2*F*, the classical realization of the formulation of this section is provided by functions $X_i, X_j, \ldots$ on the isotangent bundle $T^* \hat{E}(r, \hat{\delta}, \hat{\mathcal{R}}), \hat{\delta} = T\delta$, with local chart $a = \{p_k, r^k\}$ and isoalgebra

$$[X_i, \hat{X}_j] = \frac{\partial X_i}{\partial x^m}\, T^n_m(x, p, \ldots)\, \frac{\partial X_j}{\partial p_n} - \frac{\partial X_j}{\partial x^m}\, T^n_m(x, p, \ldots)\, \frac{\partial X_i}{\partial p_n} = \hat{C}^k_{ij}\, T^m_k X_m \ldots \quad (3.19)$$

As outlined in Sect. 2.*G*, the operator realization is given by operators X_k on an isohilbert space with a given isounit $\hat{I} = T^{-1}$ and isoalgebra

$$[X_i, \hat{X}_j] = X_i\, T\, X_j - X_j\, T\, X_i = \hat{C}^k_{ij}\, T X_k. \quad \ldots (3.20)$$

The unique and unambiguous map interconnection realizations (3.19) and (3.20) is the isoquantization of Sect. 2*G*.

3.C. Isotopies and isodualities of Lie groups.

A *right Lie-Santilli group* $\hat{G}$ (See [47], [49], [61], [62] for original studies and monographs [3], [24], [31], [74] with large number of quoted papers for independent studies) on an isospace $\hat{S}(x, \hat{F})$ over an isofield $\hat{F}, \hat{I} = T^{-i}$ (of isoreal $\hat{\mathcal{R}}$ or isocomplex numbers $\hat{C}$), also called *isotransformation group of isogroup*, is a group which maps each element $x \in \hat{S}(x, \hat{F})$ into a new element $x' \in \hat{S}(x, \hat{F})$ via the isotransformations $x' = \hat{U} * x = \hat{U}Tx$, T fixed, such that : (1) The map $(U, x) \to \hat{U} * x$ of $\hat{G} \times \hat{S}(x, \hat{F})$ onto $\hat{S}(x, \hat{F})$ is isodifferentiable (2) $\hat{I} * \hat{U} = \hat{U} * \hat{I} = \hat{U} \quad \forall\ \hat{U} \in \hat{G}$; and (3) $\hat{U}_1 * (\hat{U}_2 * x) = (\hat{U}_1 * \hat{U}_2) * x, \quad \forall\ x \in \hat{S}(x, \hat{F})$ and $\hat{U}_1, \hat{U}_2 \in \hat{G}$. *A left isotransformation group is defined accordingly.*

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The notions of *connected or simply connected transformation groups* carry over to the isogroups in their entirety. We consider hereon the connected isotransformation groups. Right or left isogropus are characterized by the following laws [47].

$$\hat{U}(0) = \hat{I},\ \hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w}') * \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}'),\ \hat{U}(\hat{w}) * \hat{U}(-\hat{w}) = \hat{I},\ \hat{w} \in \hat{F}. \qquad \ldots (3.21)$$

Their most direct realization of the isotransformation groups is that via isoexponentiation (3.3),

$$\hat{U}(\hat{w}) = \prod_k e_{\hat{\xi}}^{i\hat{w}_k * X_k} = \prod_k e_{\xi}^{i} X_k * \hat{w}_k = \hat{I}\{\prod_k e^{iw_k TX_k}\} = \prod_k e^{iX_k Tw_k}\hat{I} \qquad \ldots (3.22)$$

where the X's and w's are the infinitesimal generator and parameters, respectively, of the original algebra L. Equations (3.22) hold for some open neighbourhood of N of the isoorigin of $\hat{L}$ and, in this way, characterize some open neighbourhood of the isounit of $\hat{G}$. Then the isotransformations can be reduced to an ordinary transform for computational convenience,

$$x' = \hat{U} * x = \{\prod_k e_{\hat{\xi}}^{iX_k * w_k}\} * x = \{\prod_k e^{iX_k Tw_k}\}x, \qquad \ldots (3.23)$$

with the understanding that, on rigorous mathematical grounds, only the isotransform is correct.

Still another important result obtained in [47] is the proof that conventional group composition laws admit a consistent isotopic lifting, resulting in the following isotopy of the Baker-Campbell-Hausdorff Theorem

$$\{e_{\xi}^{X}\} * (e_{\xi}^{X}\} = e_{\xi}^{X_3},\ X_3 = X_1 + X_2 + [X_1, \hat{,} X_2]/2 + [(X_1 - X_2), \hat{,} [X_1, \hat{,} X_2]]/12 + \ldots \qquad \ldots (3.24)$$

Note the crucial appearance of the isotopic element $T(x, \dot{x}, \ddot{x}, \ldots)$ in the exponent of the isogroup. This ensures a structural generalization of Lie's theory of the desired nonlinear, nonlocal and noncanonical form. For details see [49] and [74].

The structure theory of isogroups is also vastly unexplored at this writing. In the following we shall point out that the conventional structure theory of Lie groups does indeed admit a consistent isotopic lifting. The isotopies of the notions of weak and strong continuity of [22] are a necessary pre-requisite. Let $\hat{L}$ be a (finite-dimensional) Lie-Santilli algebra with (ordered) basis $\{X_k\}$, $k = 1, 2, \ldots, N$. For a sufficiently small neighborhood N of the isoorigin of $\hat{L}$, a generic element of $\hat{G}$ can be written

$$\hat{U}(w) = \prod^{*}_{k=1,2,\ldots,N} e_{\hat{\xi}}^{iX_k w_k}, \qquad \ldots (3.25)$$

which characterizes some open neighborhood M of the isounit $\hat{I}$ of $\hat{G}$ The map

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$$\hat{\Phi}_{\hat{U}_1}(\hat{U}_2) = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-\hat{I}}, \qquad \ldots (3.26)$$

for a fixed $\hat{U}_1 \in \hat{G}$, characterizes an *inner isoautomorphism* of $\hat{G}$ onto $\hat{G}$. The corresponding isoautomorphism of the algebra $\hat{L}$ can be readily computed by considering the above expression in the neighborhood of the isounit $\hat{I}$. In fact, we have

$$\hat{U}'_2 = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-\hat{I}} \simeq \hat{U}_2 + w_1 w_2 [X_2, X_1] + O^{(2)}. \qquad \ldots (3.27)$$

The reduction of the isogroups to isoalgebras requires the knowledge of isodifferentials $\hat{d}w = Tdw$ and isoderivatives $\hat{d}/\hat{d}w = \hat{I}dw$, under which we have the following expression in one dimension :

$$i^{-1} \frac{\hat{d}}{\hat{d}w} \hat{U} \mid_{w=0} = X * e_{\xi}^{iwX} \mid_{w=0} = X. \qquad \ldots (3.28)$$

where we have used the isodifferential $\hat{d}w_k = T_k{}^i\, dw_i$ and related isoderivative (Sect. 2.C).

Thus, to every inner isoautomorphism of $\hat{G}$, there corresponds an inner isoautomorphism of $\hat{L}$ which can be expressed in the form :

$$(\hat{L})_i^j = \tilde{C}_{ki}^j\, w^k. \qquad \ldots (3.29)$$

The isogroup $\hat{G}_a$ of all inner isoautomorphism of $\hat{G}$ is called the isoadjoint group. It is possible to prove that the Lie-Santilli algebra of $\hat{G}_a$ is the isoadjoint algebra $\hat{L}_a$ of $\hat{L}$. This establishes that the connections between algebras and groups carry over in their entirety under isotopies.

We mentioned before that the direct of isoalgebras is the conventional operation because the addition is not lifted under isotopies (otherwise there will be the loss of distributivity, see [59]. The corresponding operation for groups is the semidirect product which, as such, demands care in its formulation.

Let $\hat{G}$ be an isogroup and $\hat{G}_a$ the group of all its inner isoautomorphisms. Let $\hat{G}_a^0$ be a subgroup of $\hat{G}_a$, and let $\hat{\Lambda}(\hat{G})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}_a^0$. The semidirect isoproduct $\hat{G} \hat{\times} \hat{G}_a^0$ of $\hat{G}$ and $\hat{G}_a^0$ and is the isogroup of all ordered pairs

$$(\hat{g}, \hat{\Lambda}) * (g', \hat{\Lambda}) = (\hat{g} * \hat{\Lambda}(\hat{g}'), \hat{\Lambda} * \hat{\Lambda}'), \qquad \ldots (3.30)$$

with total isounit given by $(\hat{I}, \hat{I}_{\hat{\Lambda}})$ and invese $\hat{g}, \hat{\Lambda}s)^{-\hat{1}}$ $= (\hat{\Lambda}^{-\hat{1}} = (\hat{\Lambda}^{-\hat{1}}\, (\hat{g}^{-\hat{1}}), \hat{\Lambda}^{-1}$. The above notion plays an important role in the isotopies of the inhomogeneous space-time symmetries outlined later on.

Let $\hat{G}_1$ and $\hat{G}_2$ be two isogroups with respective isounits $\hat{I}_1$ and $\hat{I}_2$. The *direct isoproduct* $\hat{G}_1 \hat{\otimes} \hat{G}_2$ of $\hat{G}_1$ and $\hat{G}_1$ is the

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isogroup of all ordered pairs $(\hat{g}_1, \hat{g}_2), \hat{g}_1 \in \hat{G}_1, \hat{g}_2 \in \hat{G}_2$, with isomultiplication

$$(\hat{g}_1, \hat{g}_2) * (\hat{g}'_1, \hat{g}'_2) = (\hat{g}_1 * \hat{g}'_1, \hat{g}_2 * \hat{g}'_2), \qquad \ldots (3.31)$$

total isounit $(\hat{I}_1, \hat{I}_2)$ and inverse $(\hat{g}_1^{-\hat{I}}, \hat{g}_2^{-\hat{I}})$, The isotopies of the remaining aspects of the structure theory of Lie groups can then be investigated by the interested reader.

Let $\hat{G}$ be an N-dimensional isotransformation group of Class I with infinitesimal generators X_k, $k = 1, 2, .., N$. The isodual Lie-Santilli group $\hat{G}^d$ of $\hat{G}$ ([52], [53]) is the N-dimensional isogroup with generators $X_k^d = -X_k$ constructed with respect to the isodual isounit $\hat{I}^d = -\hat{I}$ over the isodual isofield $\hat{F}^d$. By recalling that $w \in F \Rightarrow w^d \in F^d, w^d = -w$, a generic element of $\hat{G}^d$ in a suitable neighborhood of $\hat{I}^d$ is therefore given by

$$\hat{U}^d(\hat{w}^d) = e_{\xi}^{i^d \hat{w}^d} * X^d = -e_{\xi}^{i\hat{w}} * X = -\hat{U}(\hat{w}). \qquad \ldots (3.32)$$

The above antiautomorphic conjugation can also be defined for conventional Lie group, yielding the *isodual Lie group* G^d of G with generic elements $U^d(w^d) = e_{\xi}^{iw^d X} = -e_{\xi}^{iwX}$.

The symmetries significant for this paer are the following ones: the conventional form G, its isodual G^d, the isotopic form $\hat{G}$ and the isodual isotopic form $\hat{G}^d$. These different forms are useful for the respective characterization of particles and antiparticles in vacuum (exterior problem) or within physical media (interior problem).

It is hoped that the reader can see from the above elements that the entire conventional Lie's theory does indeed admit a consistent and nontrivial lifting into the covering Lie-Santilli formulation. Particularly important are the isotopies of the conventional representation theory, known as the *isorepresentation theory*, which naturaly yields the most general known, nonlinear, nonlocal and noncanonial representations of Lie groups. Studies along these latter lines were initiated by Santilli with the isorepresentations of $S\hat{U}(2)$ and of $S\hat{U}(3)$ [61], by Klimyk and Santilli Klimyk [27], and others.

As reviewed in Sect. 2.*F*, a classical realization of the formulation of this section is given by the isotangent bundle $T^* \hat{E}(r, \hat{\delta}, \hat{R}, \hat{\delta} = T\delta$, with local chart and isounit

$$a = \{r^k, p_k\}, \mu = 1, 2, 3, 4, 5, 6, k = 1, 2, 3, \quad \hat{I}_2 = \text{diag.}\,(\hat{1}, \hat{1}). \qquad (3.33)$$

the Hamilton-Santilli equations

$$\hat{d}a^{\mu}/\hat{d}t = w^{\mu a}\, T_{2a}^{\nu}\, \frac{\partial H}{\partial a^{\nu}}, \qquad \ldots (3.34)$$

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where $w^{\mu\alpha}$ is the familiar canonical Lie tensor and their exponentiated form

$$a(t) = (e^{tw^{\mu\alpha}T^{\nu}_{2\alpha}(\partial H/\partial a^{\mu})^{\partial/\partial a^{\alpha}}}\} a(0), \qquad \ldots (3.35)$$

where we have ignored the factorization of the isounit in the isoexponent for simplicity.

As recalled in Sect. 2*G*, an operator realization of the Lie-Santilli isogroups is given by *isounitary transformations* $x' = \hat{U} * x$ on an isohibert space $\mathcal{H}$ with

$$\hat{U} * \hat{U}^{\dagger} = \hat{U}^{\dagger} * \hat{U} = \hat{I}, \qquad \ldots (3.36)$$

with realization in terms of an *isohermitean operator H*

$$\hat{U} = \hat{e}^{\hat{i}Ht} = \{e^{iHTt}\}\hat{I}. \qquad \ldots (3.37)$$

The above classical and operator realizations are also interconnected in a unique and unambiguous way by the isoquantzation (Sect. 2.*G*).

3.D. Santilli's fundamental theorem on isosymmetries. We are now equipped to review without proof the following important result formulated in [52] and then studied in details in [61] and [62]

Theorem 3.5. *Let G be an N-dimensional Lie group of isometries of an m-dimensional metric or pseudo-metric space* $S(x, g, F)$ *over a field F*

$$G : x' = A(w)\, x,\ (x' - y')\tau\, A\, \tau\, gA(x - y) \equiv (x - y)\tau\, g(x - y),\ A\tau\, g\, A$$
$$AgA\tau = g, \qquad \ldots (3.38)$$

Then the infinitely possible isotopies $\hat{G}$ *of G of Class III characterized by the same generators and parameters of G and new isounits* $\hat{I}$ *(isotopic elements T), automatically leave invariant the isocomposition on the isospaces* $\hat{S}(x, \hat{g}, \hat{F}), \hat{g} = Tg\, \hat{I} = T^{-1}$,

$$\hat{G} : x' = \hat{A}(w) * x,\ (x' - y')\hat{U} * \hat{A}\hat{U}\hat{g}\,\hat{A} * (x - y) = (x - y)\hat{U}\hat{g}\,(x - y),\ \hat{A}\hat{U}\hat{g}\,\hat{A}$$
$$= \hat{A}\,\hat{g}\,\hat{A}\,\hat{U} = \hat{I}\,\hat{g}\,\hat{I}, \ \ldots (3.39)$$

The "direct universal" of the resulting isosymmetries for all infinitely possible isotopies $g \to \hat{g}$ is then evident owing to the completely unrestricted functional dependence of the isotopic element T in the isometric $\hat{g} = Tg$. One should also note the insufficiency of the so-called trivial isotopy

$$X_k - X'_k = X_k\, \hat{I}. \qquad \ldots (3.40)$$

for the achievement of the desired form-invariance. In fact, inder the above mapping the isoexponentiation becomes

$$e^{iX'_k} * w_k = \{e^{iX'_k\, Tw_k}\}\hat{I} = \{e^{iX_k wk}\}\hat{I}, \qquad \ldots (3.41)$$

namely, we have the disappearance precisely of the isotopic element T in the exponent which provides the invariance of the isoseparation.

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3.E. Isotopies and isodualities of the rotational symmetry. We now illustrate the Lie- Santilli isotheory with the first mathematically and physically significant case, *the isotopies of the rotational symmetry, also called isorotational symmetry*. They were first achieved in [53] and then studied in details in [61], including the isotopies of SU(2), their isorepresentations, the iso-Clebsh-Gordon coefficients, etc.

Consider the lifting of the perfect sphere in Euclidean space $E(r, \delta, \mathcal{R})$ with local coordinates $r = (x, y, z)$, and metric $\delta = diag.\ (1, 1, 1)$ over the reals $\mathcal{R}$.

$$r^2 = r^t \delta r = xx + yy + zz, \qquad \ldots (3.42)$$

into the most general possible ellipsoid of Class III on isospace $\hat{E}^{III}(r, \hat{\delta}, \hat{\mathcal{R}})$, $\hat{\delta} = T\delta$, $T = \text{diag.}\ (g_{11} \cdot g_{22}, g_{33})$, $\hat{I} = T^{-1}$.

$$\hat{r^2} = r^t\, \hat{\delta}\, r = x g_{11}\, y + y\, g_{22}\, y + z\, g_{33} z,\ \delta^\tau = \hat{\delta},\ \hat{} g_{kk} = g_{kk}(t, r, \dot{r}, \ddot{r}, \ldots) \neq 0, \qquad \ldots (3.43)$$

The invariance of the original separation r^2 is the conventional rotational symmery $O(3)$. The isotopic techniques then permit the construction, in the needed explicit and finite form, of the isosymmetries $\hat{O}(3)$ of all infinitely possible generalized invariants $\hat{r^2}$ via the following steps: (1) Identification of the basics isotopic element T in the lifting $\delta \to \hat{\delta} = T\delta$, which, in this particular case, is given by the new metric $\hat{\delta}$ itself, $T \equiv \hat{\delta}$ and identification of the fundamental unit of the theory. $\hat{I} = T^{-1}$; (2). Consequential lifting of the basic field $\hat{\mathcal{R}}(n, +, \times) \Rightarrow \hat{\mathcal{R}}(\hat{n}, +, *)$; (3) Identification of the isospace in which the generalized metric $\hat{\delta}$ is defined, which is given by the three-dimensional isoeuclidean spaces $\hat{E}(r, \hat{\delta}, \hat{\mathcal{R}})$, $\hat{\delta} = T\delta$, $\hat{I} = T^{-1}$; (4). Construction of the $\hat{O}(3)$ symmetry via the use of the original parameters of $O(3)$ the Euler's angles θ_k, $k = 1, 2, 3$), the original generators (the angular momentum components $M_k = \epsilon_{kij} r^i p_j$) in their fundamental (adjoint) representation, and the new metric δ; and (5) Classification, interpretation and application of the results.

The explicit construction of $\hat{O}(3)$ is straightforward. According to the Lie-Santilli theory, the connected component $S\hat{O}(3)$ of $\hat{O}(3)$ is given by [53]

$$S\hat{O}(3) : r' = \hat{R}(\theta) * r,\ \hat{R}(\theta) * r,\ \hat{R}(\theta) = \prod_{k=1,2\,3}^{*} e_{\xi}^{iM_k\theta_k} = (\prod_{k=1,\,2,\,3} e^{iM_k\, T\theta_k]\hat{I}}, \qquad \ldots (3.44)$$

while the discrete component is given by the *isoinversions* [loc. cit.] $r' = \hat{\pi} * r = \pi r = -r$, where π is the conventional inversion.

Under the assumed conditions on the isotopic element T, the convergence of isoexponentiations is ensured by the original

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convergence, thus permitting the explicit construction of the isorotations, with example around the third axis [53]

$$x' = x \cos [\theta_3 (g_{11} g_{22})^{1/2}] + y\, g_{22} (g_{11} g_{22})^{-1/2} \sin [\theta_3 (g_{11} g_{22})^{1/2}],$$

$$y' = -x\, g_{11} (g_{11} g_{22})^{-1/2} \sin [\theta_3 (g_{11} g_{22})^{1/2}] + y \cos [\theta_3 [g_{11} g_{22})^{1/2}],$$

$$z' = z. \qquad \ldots (3.45)$$

(see [61b] for general isorotations). One should note that the argument of the trigonometric functions as derived via the above isoexponentiation coincides with the isoangle of the isotrigonometry in $\hat{E}(r, \delta, \hat{R})$ (see paper [60] thus confirming the remarkable compatibility and interconnections of the various branches of the isotopic theory.

The computation of the isoalgebras $\hat{o}(3)$ of $\hat{O}(3)$ is then straightforward [loc. cit.]. In fact, when M_k are assumed to be in their regular representation we have [53]

$$\hat{o}(3): [M_i, \hat{M}_j] = M_i\, T\, M_j - M_j\, T\, M_i = \hat{C}_{ij}^k * M_k,\ \hat{C}_{ij}^k = \epsilon\ ijk\ g_{kk}^{-1}\, \hat{I}. \qquad \ldots (3.46)$$

The above isoalgebra illustrates the explicit dependence of the structure *functions*. The proof of the isomorphism $\hat{o}(3) \approx o(3)$ as done [loc. cit.] via a suitable reformulation of the basis under which the structure functions recover the value $\hat{\epsilon}_{ijk} = \epsilon_{ijk}\, \hat{I}$.

The isocenter of $\hat{so}(3)$ is characterized by the *isocasimir invariants*

$$C^{(0)} = \hat{I},\ C^{(2)} = \hat{M^2} = M * M = \Sigma_{k=1,2,3}\, M_k\, T\, M_k. \qquad \ldots (3.47)$$

In hadronic mechanics the realization is the following. The linear momentum operator has the isotopic form for a diagonal isounit

$$p_k * | \hat{\psi} > = -i\, \hat{\nabla}_k | \hat{\psi} > = -i \hat{I}_k^i \nabla_i | \hat{\psi} > .$$

The fundamental isocommutation rules are then given by

$$[r^i, \hat{p}_j] = i\delta_j^i = i\hat{I}\delta_j^i,\ [r^i, \hat{r}^j] = [p_i, \hat{p}_j] = 0.$$

However, in their contravariant form the coordinates are given $r_k = \hat{\delta}_{ki}\, r^i$. As a result $\hat{\nabla}_i\, r_j = \delta_{ij}$ (where the delta is the *conventional* Kronecker delta). In this case the fundamental isocommutation rules are given by

$$[r_i, \hat{p}_j] = i\delta_j^i = i\hat{I}\delta_j^i,\ [r_i, \hat{r}_j] = [p_i, \hat{p}_j] = 0,$$

namely, their eigenvalues *coincide* with the quantum ones. The operator isoalgebra $\hat{o}(3)$ with generators $M_k = \epsilon_{kij}\, r_i p_j$

$$\hat{o}(3): [M_i, \hat{M}_j] = M_i\, T\, M_j - M_j\, T\, M_i = i\, \epsilon_{ij}^k * M_k,\ \hat{\epsilon}_{ij}^k = \epsilon_{ijk}\, \hat{I},$$

namely the *product of the algebra is generalized, but the structure consants are the conventional ones* (see [61] for details). The above

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results illustrates again the abstract identity of quantum and hadronic mechanics.

Note the nonlinear-nonlocal-noncanonical character of isotransformations (3.45) owing to the unrestricted functional dependence of the diagonal elements g_{kk}. Note also the extreme simplicity of the final results. In fact, the explicit symmetry transformations of separation (3.43) are provided by just plotting the given g_{kk} values into transformations (3.45) without any need of any additional computation. Note finally that the above invariance includes as particular case the general isosymmetry $\hat{O}(3)$ of (the space-component of) gravitation which, since it is locally Euclidean,remains isomorphic to $O(3)$.

As an example, the symmetry of the space-component of the Schwartzschild line element is given by plotting the following values

$$g_{11} = (1 - M/r)^{-1}, g_{22} = r^2, g_{33} = r^2 \sin^2 \theta, \qquad \ldots (3.48)$$

(see next section for the full (3+1)- dimensional case).

Despite this simplicity, the implications of the above results are nontrivial. On physical grounds, the isounit $\hat{I} > 0$ permits a direct representation of the nonspherical shapes, as well as all their infinitely possible deformations. By recalling that 0(3) is a *theory of rigid bodies*, $\hat{O}(3)$ results to be a *theory of deformable bodies* (53) with fundamentally novel physical applications in the theory of elasticity, nuclear, particle physics, crystallography, and other fields [61], [62].

On mathematical grounds, we have equally intriguing novel insights. To see them, one must first understand the background isogeometry $\hat{E}^{III}(r, \hat{\delta}, \hat{\mathcal{R}})$ which unifies all possible conics in $E(r, \delta, \mathcal{R})$ as mentioned earlier. To be explicit in this important point, the geometric differences between (oblate or prolate) ellipsoids and (elliptic or hyperbolic) paraboloids have mathematical sense when projected in our Euclidean space $E(r, \delta, \mathcal{R})$. However all these surfaces are geometrically unified with the perfect isosphere in $\hat{E}(r, \hat{\delta}, \hat{\mathcal{R}})$.

These geometric occurrences permits the unification of $\mathbf{O}(3)$ and $O(2.1)$, as well as of all their infinitely possible isotopes. In fact, the classification of all possible isosymmetries $\hat{O}(3)$, achieved in the original derivation [53], includes :

(1) The compact $O(3)$ symmetry evidently for $\hat{\delta} = \delta$ = diag. (1,1,1);

(2) The noncompact $O(2.1)$ symmetry evidently for $\hat{\delta}$ = diag. (1,1, – 1)

(3) The isodual $O^d(3)$ of $O(3)$ holding for $\hat{\delta}$ = diag. (– 1, – 1, – 1);

(4) The isodual $O^d(2.1)$ of $O(2.1)$ holding for $\hat{\delta}$ = diag. (– 1, – 1, 1);

(5) The infinite family of compact isotopes $\hat{O}(3) \approx O(3)$ with $I > 0$ for

$$\hat{\delta} = \text{diag.}\ (b_1^2, b_2^2, b_3^2), \quad b_k > 0;$$

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(6) The infinite family of noncompact isotopes $\hat{O}(2.1) \approx O(2.1)$ for

$$\delta = \text{diag.}\ (b_1^2, b_2^2, -b_3^2);$$

(7) The infinite family of compact isodual isotopes $\hat{O}^d(3) \approx O^d(3)$ for

$$\hat{\delta} = \text{diag.}\ (-b_1^2, -b_2^2, -b_3^2);$$

(8) The infinite family of isodual isotopes $\hat{O}^d(2.1) \approx O^d(2.1)$ for

$$\hat{\delta} = \text{diag.}\ (-b_1^2, -b_2^2, b_3^2).$$

Even greater differentiations between the Lie and Lie-Santilli theories occur in their representations because of the change in the eigenvalue equations due to the nonunitarity of the map indicated in Sect. 1, from the familiar form $H\psi = E^0\psi$, isotopic form $H^*\hat{\psi} = \hat{E} * \hat{\psi} \equiv E\hat{\psi}, E^0 \neq E$), thus implying generalized weights, Cartan tensors and other structures studied earlier

The first differences emerge in the spectrum of eigenvalues of $\hat{O}(2)$ and $O(2)$. in fact, the $o(2)$ albegra on a conventional Hilbert space solely admits the spectrum $M = 0, 1, 2, 3$ (as a necessary condition of unitarity). For the covering $\hat{O}(2)$ isoalgebra on an isohilbert space with isotopic element $T = \text{Diag.}\ (g_{11}, g_{22})$, the spectrum is instead given $\hat{M} = g_{11}^{-1/2} g_{22}^{-1/2} M$ and, as such, it can acquire *continuous* values in a way fully consistent with the condition, this time, of isounitarity. For the general $\hat{O}(3)$ case see also the detailed studied of ref. [61].

Similarly, the unitary irreducible representations of su(2) are characterize the familiar eigenvalues

$$J_3\hat{\psi} = M\psi,\ J^2\psi = J(J+1)\psi,\ M = J, J-1, ..., -J,\ J = 0, \tfrac{1}{2}, 1, ... \quad ... (3.49)$$

Three classes of irreducible isorepresentation of $\hat{su}(2)$ were identified in [63] which, for the adjoint case, are given by the following generalizations of Pauli's matrices:

(1) *Regular isopauli matrices*

$$\hat{\sigma}_1 = \Delta^{-1/2}\begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix} \hat{\sigma}_2 = \Delta^{-1/2}\begin{pmatrix} 0 & -ig_{11} \\ ig_{22} & 0 \end{pmatrix}, \hat{\sigma}_3 = \Delta^{-1/2}\begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix}$$

... (3.50a)

$$T = \text{diag.}\ (g_{11}, g_{22}),\ \Delta = \det T = g_{11}\, g_{22} > 0,\ [\hat{\sigma}_i, \hat{\sigma}_j]^{\xi} = i2\Delta^{1/2} \in_{ijk} \hat{\sigma}_k.$$

... (3.50b)

$$\hat{\sigma}_3 * |\hat{b}> \pm \Delta^{1/2} |\hat{b}>,\ \hat{\sigma}^{\hat{2}} * |\hat{b}> = 3\Delta |\hat{b}>, \quad ... (3.50c)$$

(2) *Irregular isopauli matrices*

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$$\hat{\sigma}_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \hat{\sigma}_2' = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2, \hat{\sigma}_3' = \begin{pmatrix} g_{22} & 1 \\ 1 & -g_{11} \end{pmatrix} = \Delta \hat{I} \sigma_3,$$

... (3.51a)

$$[\hat{\sigma}'_1, \hat{\sigma}'_2]_{\hat{\xi}} = 2i\hat{\sigma}'_3, \quad [\hat{\sigma}_2', \hat{\sigma}_3']_{\xi} = 2i\,\Delta\,\hat{\xi}_1', \quad [\hat{\sigma}_3', \hat{\sigma}_1']_{\hat{\xi}} = 2i\hat{\sigma}_2',$$

... (3.51b)

$$\hat{\sigma}_3' * |\hat{b}> = \pm\,\Delta\,|\hat{b}>, \quad \hat{\sigma}^{\hat{2}} * |\hat{b}> = \Delta(\Delta+2)\,|\,\hat{b}>.$$

... (3.15c)

(3) *Standard isopauli matrices*

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & \lambda \\ 0 & -\lambda \end{pmatrix},$$

... (3.52a)

$$T = \text{diag.}\,(\lambda, \lambda^{-1}),\ \lambda \neq 0, \quad \Delta = \det T = 1, \quad [\hat{\sigma}''_i, \hat{\sigma}''_j]_{\hat{\xi}} = i \in_{ijk} \hat{\sigma}''_k,$$

... (3.52)

The primary differences in the above isorepresentations are the following. For the case of the regular isorepresentations, the isotopic contributions can be factorized with respect to the conventional Lie spectrum. For the iregular case this is no longer possible. Finally, for the standard case we have conventional spectra of eigenvalues under a generalized structure of the matrix representations, as indicated by the appearance of a completely unrestricted, integro-differential function λ.

The regular and irregular representations of $\hat{o}(3)$ and $\hat{su}(2)$ are applied to the angular momentum and spin of particles under extreme physical conditions, such as an electron in the core of a collapsing star. The standard isorepresentations are applied to conventional particles evidently because of the preservation of conventional quantum numbers The appearance of the isotopic degrees of freedom then permit novel physical applications, that is, applications beyond the capacity of Lie's theory even for the simpler case of preservation of conventional spectra (see Section 3.G.)

The spectrum-preserving map from the conventional representations J_g of a Lie-algebra L with metric tensor g to the covering isorepresentations $\hat{J}_{\hat{g}}$ of the Lie-Santilli algebra $\hat{L}$ with isometric $\hat{g} = Tg$ and isounit $\hat{I} = T^{-1}$ is important for physical application. It is called the *klimyk rule* [27] and it is given by

$$\hat{J}_{\hat{g}} = J_g\, p, \quad p = k\hat{I},\ k \in \hat{F}, \qquad \text{... (3.53)}$$

under which Lie algebras are turned into Lie-Santilli isoalgebras

$$J_i J_j - J_j J_i = C_{ij}^k J_k \equiv (\hat{J}_i * \hat{J}_j - \hat{J}_j * \hat{J}_i) k^{-1} T = C_{ij}^k k^{-1} T \hat{J}_k,$$

that is,

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$$\hat{J}_i * \hat{J}_j - \hat{J}_j * \hat{J}_i = C_{ij}^k \hat{J}_k,$$

thus showing the preservation of the original structure constants.

However, by no means, the Klimyk rule can produce *all* Lie-Santil isoalgebras, because the latter are generally characterized by nounitary transforms of conventional algebras, with a general variation of the structure constants.

Nevertheless, the Klimyk rule is sufficient for a number of physical applications where the preservation of conventional quantum numbers is important, because it permits the identification of one specific and explicit form of standard isorepresentations with "hidden" degrees of freedom represented by the isotopic element T available for specific uses. For instance, the standard isopauli matrices permit the reconstruction of the exact isospin symmetry in nuclear physics under electromagnetic and weak interactions [63], or the construction of the isoquark theory with all conventional quantum numbers, yet an *exact confinement* (with an identically null probability of tunnel effects for free quarks because of the incoherence between the interior and exterior Hilbert spaces) [68], and other novel applications.

3.F. Isotopies and isodualities of the Lorentz and Poincare' symmetries. Consider the line element in Minkowski space $x^2 = x^\mu \eta_{\mu\nu} x^\nu$, $\mu, \nu = 1, 2, 3, 4$, with local coordinates $x = \{x^1, x^2, x^3, x^4\}$, $x^4 = c_0 t$, and metric $\eta =$ diag. $(1, 1, 1, -1)$. Its simple invariance group, the six-dimentsional Lorentz group L (3.1), is characterized by the (ordered sets of) parameters given by the Euler's angles and speed parameter, $w = \{w_k\} = \{\theta, v\}$, $k = 1, 2, ..., 6$, and generators $X = \{X_k\} = \{M_{\mu\nu}\}$ in their known fundamental representation (see, e.g., [31], [32].

Suppose now that the Minkowskian line element is lifted into the most general possible nonlinear-integral form verifying the conditions of Class III

$$\hat{x^2} = x^\mu \hat{g}_{\mu\nu} (x, \dot{x}, \ddot{x}, ...) x^\nu, \quad \det \hat{g} \neq 0, \quad \hat{g} = \hat{g}^{\hat{U}}, \qquad ... (3.54)$$

which represent: all modifications of the Minkowski metric as encountered, e.g., in particle physics; conventional exterior gravitational line elements with $\hat{g} = \hat{g}(x)$, such as the full Schwartzschild line element; all its possible generalizations for the interior problem; etc.

The explicit form of the simple, six-dimensional invariance of generalized line element $\hat{x^2}$ was first constructed by Santilli [51] by following the space-time version of Steps 1 to 5 of the preceding section. Step 1 is the identification of the fundamendal isotopic element T via the factorization of the Minkowski metric $\hat{g} = T\eta$ which, under the assumed conditions, can always be diagonalized into the form

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$$T = \text{diag.}\,(g_{11}, g_{22}, g_{33}, g_{44}),\ T = T^{\tau},\ \det T \neq 0. \quad \ldots (3.55)$$

The fundamental isounit of the theory is then given by $\hat{I} = T^{-1}$.

Step 2 is the lifting of the conventional numbers into the isonumbers via the isofields $\hat{\mathcal{R}}(\hat{n}, +, *)$, $\hat{n} = n\hat{I}$ (which are different than those of $\hat{O}(3)$ because of the different dimension of the isounit).

Step 3 is the construction of the isospaces in which the isometric $\hat{g}$ is properly defined, which are given by the isominkowski spaces $\hat{M}(x, \hat{g}, \hat{\mathcal{R}})$. The reader should keep in mind that, when $\hat{g}$ is a conventional Riemannian metric, isospaces $\hat{M}(x, \hat{g}, \hat{\mathcal{R}})$ are not the Riemannain spaces $R(x, \hat{g}, \mathcal{R})$ because the basic units of the two spaces are different.

Step 4 is also straightforward. The *Lorentz-Santilli isosymmetry* $\hat{L}(3.1)$ is characterized by the isotransformations

$$\hat{O}(3.1)\ .\quad x' = \hat{\Lambda}(\hat{w}) * x = \Lambda(w)x, \quad \ldots (3.56)$$

verifying the basic properties

$$\hat{\Lambda}^{\tau} = \hat{\Lambda}\,\hat{g}\,\hat{\Lambda}^{\tau} = \hat{I}\hat{g}\hat{I},\ \textit{or}\ \ \Lambda\,\tau\,\hat{g}\,\Lambda = \Lambda\,\hat{g}\,\Lambda\tau = \hat{g}, \quad \ldots (3.57a)$$

$$\hat{\text{Det}}\,\hat{\Lambda} = [\text{Det}\,(\hat{\Lambda}\,T] = \pm\hat{I}. \quad \ldots (3.57b)$$

It is easy to see that $\hat{L}(3.1)$ preserves the original connectivity properties of $L(3.1)$ (see [61] for a detailed study). The connected component $S\hat{O}(3.1)$ of $\hat{L}(3.1)$ is characterized by $\hat{\text{Det}}\,\hat{\Lambda} = +\hat{I}$ and has the structure [loc. cit.]

$$\hat{\Lambda}(w) = \hat{\Pi}^{*}_{k=1,2,\ldots,6}\, e^{i\hat{x}^{\dagger}_{k}\hat{w}_{k}} = \{\prod_{k=1,2,\ldots,6} e^{iX_{k}\,Tw_{k}}\}\hat{I} \quad \ldots (3.58)$$

where the paramenters are the conventional ones, the generators X_k are also the conventional ones in their fundamental representation and the isotopic element T is given by Equations (3.23). The discrete part of $\hat{L}(3.1)$ is characterized by $\hat{\text{Det}}\,\hat{\Lambda} = -\hat{I}$, and it is given by the space-time isoinversions [loc . cit.]

$$\hat{\pi} * x = \pi\, x = -r, x^4),\quad \hat{T} * x = Tx = (r, -x^4). \quad \ldots (3.59)$$

Again, under the assumed conditions for T, the convergence of infinite series (3.58) is ensured by the original convergence, thus permitting the explicit calculation of the symmetry transformations in the needed explicit, finite form. Their space components have been given in the preceding Section 3.E. The additional *Lorentz-Santilli* isoboosts can also be explicitly computed, yielding the expression for all possible isometrics $\hat{g}$[51]

$$x'^{1} = x^{1},\ x'^{2} = x^{2}, \quad \ldots (3.60a)$$

$$x'^{3} = x^{3}\cosh\,[v\,(g_{33}\,g_{44})^{1/2}] - x^{4}\,g_{44}(g_{33}\,g_{44})^{-1/2}\sinh\,[v(g_{33}\,g_{44})^{1/2}] =$$
$$= \hat{\gamma}(x^{3} - g_{33}^{-1/2}\,g_{44}^{1/2}\,\hat{\beta}x^{4}), \quad \ldots (360b)$$

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$$x'4 = -x^3 g_{33}(g_{33}\, g_{44})^{-1/2} \sinh [v\,(g_{33}\, g_{44})^{1/2}] + x^4 \cosh [v\,(g_{33}\, g_{44})^{1/2}] =$$
$$= \hat{\gamma}\,(x^4 - g_{33}^{1/2}\, g_{44}^{-1/2}\, \hat{\beta}\, x^3), \qquad \ldots (3.60c)$$

where

$$x^4 = c_0 t,\ \beta = v/c_0,\ \beta = v^k\, g_{kk}\, v^k / c_0\, g_{44}\, c_0, \qquad \ldots (3.61a)$$

$$\cosh [v(g_{33}\, g_{44})^{1/2}] = \hat{\gamma} = |\,1 - \hat{\beta}^2\,|^{-1/2},\ \sinh [v(g_{33}\, g_{44})^{1/2}] = \hat{\beta}\hat{\gamma}. \qquad \ldots (3.61b)$$

Again, one should note : (A) the unrestricted character of the functional dependence of the isometric $\hat{g}$; (*B*) the remarkable simplicity of the final results whereby the explicit symmetry transformations are merely given by plotting the values $g_{\mu\mu}$ in Equations (3.60); and (C) the generally nonlinear-nonlocal-noncanonical character of the isosymmetry.

The isocommutation rules when the generators $M_{\mu\nu}$ are in their regular representation can also be readily computed and are given by [loc. cit]

$$\hat{O}(3.1) : [M_{\mu\nu}, \hat{M}_{\alpha\beta}] = \hat{g}_{\nu\alpha}\, M_{\beta\mu} - \hat{g}_{\mu\alpha}\, M_{\beta\nu} - \hat{g}_{\nu\beta}\, M_{\alpha\mu} + \hat{g}_{\mu\beta}\, M_{\alpha\nu}, \qquad \ldots (3.62)$$

with isocasimirs

$$C^{(0)} = \hat{I},\ C^{(1)} = \tfrac{1}{2} M_{\mu\nu}\, T\, M^{\mu\nu} = M * M - N * N, \qquad \ldots (3.63a)$$

$$C^{(3)} = \tfrac{1}{2} \in^{\mu\nu\rho\sigma} M_{\mu\nu}\, T\, M_{\rho\sigma} = -M * N,\ M = \{M_{12}, M_{23}, M_{31}\},\ N = \{M_{01}, M_{02}, M_{03}\} \qquad \ldots (3.63c)$$

The classification of all possible isotopes $S\hat{O}(3.2)$ was also done in the original construction [51] via the realizations of the isotopic element

$$T = \text{diag.}\ (\pm b_1^2, \pm b_2^2, \pm b_3^2, \pm b_4^2),\ \ b_\mu > 0,\ \mu = 1, 2, 3, 4, \qquad \ldots (3.64)$$

where the *b's* are the characteristic functions of the interior medium, resulting in:

(1) The conventional orthogonal symmery $SO(4)$ for $T = \text{diag.}\ (1, 1, 1, -1)$;

(2) The conventional Lorenty symmery $SO(3.1)$ for $T = \text{diag.}\ (1, 1, 1, 1)$;

(3) the conventional de Sitter symmetry $SO(2.2)$ for $T = \text{diag.}\ (1, 1, -1, 1)$;

(4) the isodual $SO^d(4)$ for $T = \text{diag.}\ (-1, -1, -1, 1)$;

(5) the isodual $O^d(3.1)$ for $T = -\text{diag.}\ (1, 1, 1, 1)$;

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(6) the isodual $SO^d(2.2)$ for $T = \text{diag.}\ (-1, -1, 1, -1)$;

(7) the infinite family of isotopes $S\hat{O}(4) \approx SO(4)$ for

$$T = \text{diag.}\ (b_1^2, b_2^2, b_3^2, -b_4^2);$$

(8) the infinite family of isotopes $S\hat{O}(3.1) \approx SO(3.1)$ for

$$T = \text{diag.}\ (b_1^2, b_2^2, b_3^2, b_4^2);$$

(9) the infinite family of isotopes $S\hat{O}(2.2) \approx SO(2.2)$ for

$$T = \text{diag.}\ (-b_1^2, b_2, b_3^2, b_4^2);$$

(10) the infinite family isoduals $S\hat{O}^d(4) \approx SO^d(4)$ for

$$T = \text{diag.}\ (-b_1^2, -b_2^2, -b_3^2, b_4^2);$$

(11) the infinite family of isoduals $S\hat{O}^d(3.1) \approx SO(3.1)$ for

$$T = -\text{diag.}\ (b_1^2, -b_2^2, -b_3^2, b_4^2);$$

(12) the infinite family of isoduals $S\hat{O}^d(2.2) \approx SO^d(2.2)$ for

$$T = \text{diag.}\ (b_1^2, -b_2^2, -b_3^2, -b_4^2).$$

On the basis of the above results, Santilli [61] submitted the *conjecture that all simple Lie algebra of the same dimension over a field of characteristic zero in Cartan classification can be unified into one single abstract isotopic algebra of the same dimension.*

The above conjecture was proved by Santilli for the cases $n = 3$ and 6. A theorem unifying all possible fields into the isoreals was proved by Kadeisvili et al [26] in the expectation of such general unification, but its study has remained unexplored at this writing.

In the above presentation we have shown that the lifting of the Lorentz symmetry can be naturally formulated for Class III. Nevertheless, whenever dealing with physical applications, the isotopic element is restricted to have the positive-or negative- definite structure $T = \pm\,\text{diag.}\ (b_1^2, b_2^2, b_3^2, b_4^2)$, thus restricting the isotopies $S\hat{O}(3.1) \approx SO(3.1)$ and $S\hat{O}^d(3.1) \approx SO^d(3.1)$.

The operator realization of the latter Lorentz-Santilli is the following. The linear four-momentum admits the isotopic realization

$$p_\mu * \mid \hat{\psi} > = -i\,\hat{\partial}_\mu \mid \hat{\psi} > = -iT_\mu^\nu\,\partial\nu \mid \hat{\psi} > .$$

Also for $x_\mu = \eta_{\mu\nu} x^\nu$ (where η is the conventional Minkowski metric). one can show that $\hat{\partial}_\mu x_\nu = \hat{\eta}_{\mu\nu}$. The fundamental relativistic isocommutation rules are then given by [61], [65]

$$[x_\mu, \hat{p}_\nu] = i\,\hat{\eta}_{\mu\nu},\ \ [x_\mu, \hat{x}_\nu] = [p_\nu, \hat{p}_\nu] = 0.$$

The isocommutation rules are then given by

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$$\hat{O}(3.1) : [M_{\mu\nu}, \hat{M}_{\alpha\beta}] = i(\hat{\eta}_{\nu\alpha} M_{\beta\mu} - \hat{\eta}_{\mu\alpha} M_{\beta\nu} - \hat{\eta}_{\nu\beta} M_{\alpha\mu} + \hat{\eta}_{\mu\beta} M_{\alpha\nu}), \quad \text{... (3.62)}$$

thus confirming the isomorphism $S\hat{O}(3.1) \approx SO(3.1)$ for all positive-definite T.

The Poincare'-Santilli isosymmetry

$$\hat{p}(3.1) = \hat{L}(3.1) \times \hat{T}(3.1), \quad \text{... (3.65)}$$

and its isodual $\hat{p}^d(3.1)$ have been constructed in their classical [62] and operator [62] forms as well as in their isospinorial form $\hat{\mathcal{P}}(3.1) = S\hat{L}(2.\hat{C}) \times \hat{T}(3.1)$ [69]. We here limit ourselves to a brief outline of the nonspinorial case mainly to illustrate the advances in the structure of isoalgebras and isogroups studied in this paper.

A generic element of $\hat{P}(3.1)$ can be written $\hat{A} = (\hat{\Lambda}, \hat{a})$, $\hat{\Lambda} \in \hat{O}(3.1)$, $\hat{a} \in \hat{T}(3.1)$ with isocomposition

$$\hat{A}' * \hat{A} = (\hat{\Lambda}', \hat{a}') * \hat{\Lambda}, \hat{a}) = (\hat{\Lambda} * \hat{\Lambda}', \hat{a} + \hat{\Lambda}' * a'), \quad (3.66)$$

The realization important for physical application is that via conventional generators in their adjoint representation for a system of n particles of non-null mass m_a

$$X = \{X_k\} = \{M_{\mu\nu} = \Sigma_m (x_{a\mu} p_{a\nu} - x_{a\nu} p_{a\nu}\}, p = \Sigma_a p_a\}, k = 1, 2, ..., 10, \quad \text{... (3.67)}$$

and conventional parameters $w = \{w_k\} = \{v, \theta, a\}$, where v represents the Lorentz parameters, θ represents the Euler's angles, and a characterizes conventional space-time translations.

The connected component of the isopoincare' group is given by

$$\hat{p}(3.1) : x' = \hat{A} * x, \hat{A} = \prod_k^* e_{\xi}^{iX_k w_k} = \{\prod_k e^{iX_k T w_k}\}\hat{I}, \quad \text{... (3.68)}$$

where the isotopic element T and the Lorentz generators $M_{\nu\mu}$ have the same realization as for $\hat{O}(3.1)$. The primary different with isosymmetries $\hat{O}((3.1)$ is the appearance of the isotranslations.

$$\hat{T}(3.1) * x = \{e_{\xi}^{iP\,\eta a}\} * x = e_{\xi}^{iP\hat{g}a}\} * x = x + \hat{a}, \ \hat{T}(3.1) * p \equiv 0. \quad \text{... (3.69)}$$

The general Poincare'-Santilli isotransformations are then given by [61], [62]

$$x' = \hat{\Lambda} * x \text{ Lorentz-Santilli isotransforms,} \quad \text{... (3.70a)}$$

$$x' = x + \hat{a} = x + a_0 B(s, x, \dot{x}, \ddot{x}, ...), \text{ isotranslations,} \quad \text{... (3.70b)}$$

$$x' = \Lambda x' = \hat{\pi}_r * = (-r, x^4), \text{ space isoinversions,} \quad \text{... (3.70c)}$$

$$x' = \hat{\pi}_t * x = (r, -x^4), \text{ time isoinversions,} \quad \text{... (3.70d)}$$

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where the B-functions are given by the expansions

$$B_\mu = b_\mu + a^\alpha [b_\mu, \hat{P}\alpha[/1! + a^\alpha a^\beta [[b_\mu, \hat{P}_\alpha], \hat{P}_\beta]/2! + \ldots \quad \ldots (3.71)$$

The isocommutation rules of $\hat{P}(3.1)$ in the operator realizations indicated earlier are

$$[M_{\mu\nu}, \hat{M}_{\alpha\beta}] = i(\hat{\eta}_{\nu\alpha} M_{\beta\mu} - \hat{\eta}_{\mu\alpha} M_{\beta\nu} - \hat{\eta}_{\nu\beta} M_{\alpha\mu} + \hat{\eta}_{\mu\beta} M_{\alpha\nu}),$$

$$\ldots (3.72a)$$

$$[M_{\mu\nu}, \hat{P}_\alpha] = i(\hat{\eta}_{\mu\alpha} P_\nu - \hat{\eta}_{\nu\alpha} P_\mu), [P_m, \hat{P}_\nu] = 0, \mu, \nu, \alpha, \beta = 1, 2, 3, 4,$$

$$\ldots (3.72b)$$

and the isocenter is characterized by the isocasimirs

$$C^{(0)} = \hat{I}, \; C^{(1)} = \hat{p^2} = P\,T\,P = P_\mu \hat{g}^{\mu\nu} P_\nu, \quad \ldots (3.73a)$$

$$C^{(2)} = \hat{W}^{\hat{2}} = \hat{W}_\mu \hat{g}^{\mu\nu} \hat{W}_\nu, \; \hat{W}_\mu = \varepsilon_{\mu\alpha\beta\rho} \hat{J}^{\alpha\beta} * P^\rho. \quad \ldots (3.73b)$$

The *restricted isotransformations* occur when the isotopic element T is constant.

An important application of the isotranslation is the characterization of the so-called *isoplane-waves on* $\hat{M}(x, \hat{\eta}, \hat{\mathcal{R}})$

$$\hat{\psi}(x) = e^{\frac{i}{\xi}px} = \hat{I} e^{ipTx} = \hat{I}\, e^{ip\mu} \hat{g}^{\mu\nu} x_\nu = \hat{I} e^{i(p_k b_k^2 x_k - p_4 b_4^2 x_4)}, \quad \ldots (3.74)$$

which are solutions of the isotopic field equations, represents electromagnetic waves propagating within inhomogeneous and anisotropic media such as out atmosphere and offer quite intriguing predictions for experimentally verifiable <novel> effects, that is, effects beyond the predictive or descriptive capacities of the Poincare' symmetry (see the companion paper [60]).

As one can see, the verification of total convservation laws (for a system assumed as isolated from the rest of the universe), is intrinsic in the very structure of the isosymmetry. In fact, the generators are the conventional ones and, since they are invariant under the action of the group they generate, they characterize conventional total consservation laws. The simplicity of reading off the total conservation laws from the generators of the isosymmetry should be compared with the rather complex proof in conventional gravitational theories.

The isodual *Poincare'-Santilli isosymmetry* $\hat{P}^d(3.1)$ is characterized by the isodual generators $X_k^d = -X^k$, the isodual parameters $w_k^d = -w_k$, and the isodual isotopic element $T^d = -T$, resulting in the change of sign of isotransforms. This impies a novel *law of universal invariant under isoduality* which essentially state that any system which is invariant under a given symmetry is automatically invariant under its isodual. In turn, this law apparently permits novel advances in the study of antiparticles [61].

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The significance of the Lie-santilli isotheory for gravitation is illustated by the following important property of the isosymmetry $\hat{P}(3.1)$ which evidently follows from of Theorem 3.5:

Theorem 3.6 [51]. *The Poincare'-Santilli isosymmetry* $\hat{P}(3.1)$ *is directly universal for all infinitely possible (3+1) - dimensional invariants*

$$(x-y)^{\mu}\,\hat{\eta}_{\mu\nu}\,(x,\dot{x},\ddot{x},...)\,(x-y)^{\nu},\ \hat{\eta}=T\eta, \qquad ...\ (3.75)$$

Note that the above theorem includes as particular cases the conventional Riemannian metric $g(x)=\hat{\eta}(x)$, thus providing the universal invariance of *exterior* gravitation in vacuum. More generally, the theorem includes all infinitely possible signature-preserving modifications of the Minkowski and Riemannian metrics for interior problems. The simplcity of this universal invariance should also be kept in mind and compared with the known complexity of other approaches to nonlinear symmetries. In fact, one merely *plots* the $g_{\mu\mu}$ elements in isotransforms (3.45), (3.60), (3.70) without any need to compute anything, because the invariance of general separation (3.75) is ensured by the theorem. For numerous examples, see [61], [62].

As anticipated in Sect. *I.E.* a remarkable property of the Lie-Santilli theory is the capability to unify in one, single, abstract isosymmetry $\hat{P}(3.1)$ all possible linear or nonlinear, local or nonlocal, Hamiltonian or nonhamiltonian, relativistic or gravitational, exterior and interior, classical and operator systems.

3.G : Mathematical and physical applications. Lie's theory is known to be at the foundation of virtually all branches of mathematics. The existence of intriguing and novel applications in mathematics originating from the Lie-Santilli theory is then self-evident.

With the understanding that mathematical studies are at their first infancy, the isotopies have already identified new branches of mathematics besides isoalgebras, isogroups and isorepresentations. We here mention: the new branch of number theory dealing with isonumbers; the new branch of functional isoanalysis dealing with T-operator special isofunctions, isotransforms and isodistribution; the new branch of topology dealing with the peculiar integro-differential topology of the isotopic theory; the new branch of the theory of manifold dealing with isomanifolds and their intriguing properties; and so on. It is hoped that interested mathematicians will contribute to these novel mathematical advances which have been identified and developed until now solely by physicists.

Lie's theory in its traditional linear-local-canonical formulation is also known to be at the foundation of all branches of contemporary physics. Profound physical implications due to the covering, nonlinear-nonlocal-noncanonical Lie-Santilli theory cannot therefore be dismissed in a credible way.

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With the understanding that these latter applications too are at the beginning and so much remains to be done, let us recall the following applications of the Poincare'-Santilli isosymmetry $\hat{P}(3.1)$ (see [61] and [62] for details):

(1) The universal invariance of all possible *conventional* gravitation [51].

(2) The geometric unification of the special and general relativities. In fact, the abstract isotope $\hat{P}(3.1)$ unifies the isosymmetry with gravitational isounit $\hat{I} = [T(x)]^{-1}$, $g(x) = T(x)\eta$, and the realization with isounit $I = \text{diag.}\,(1, 1, 1, 1)$ characterizing the special relativity [51].

(3) The universal invariance for all possible interior extensions of relativistic and gravitational theories [51].

(4) Reconstruction at the isotopic level of the *exact* SU(2)- isospin *symmetry under electromagnetic and weak interactions via* the use of the standard isopauli matrices (3.52) with $\lambda^2 = m_p/m_n$ [63];

(5) Quantitative representation of Rauch's interferometric measures on the 4π –spinorial symmetry via the isotopies of Dirac's equation invariant under $\hat{P}(3.1)$ [69];

(6) First numerical representation of the total magnetic moment of few-body nuclei via the $\hat{SO}(3)$ symmetry and its direct representation of the deformation of the charge distribution of nucleous and consequential alteration of their intrinsic magnetic moments [69];

(7) Nonlocal representation of the Bose-Einstein correlation from first isotopic principles in full numerical agreement with the data from the UAI experiments, while permitting a causal description of nonlocal interactions and the reconstruction of their exact Poincare symmetry at the isotopic level ([58], [8]);

(8) Quantitative representation of the electron pairing in superconductivity [1];

(9) Quantitative-numerical representation of the behaviour of the meanlives of unstable hadrons with speed (which, as well know, are anomalous between 30 and 100 *GeV* and conventional between 100 and 400 *GeV* for the K^0-system) via the isominkowskian geometrization of the physical medium in their interior ([6], [7]);

(10) Application to quarks theories via Klimyk rule for the standard isorepresentations of $\hat{SO}(3)$ with conventional quantum numbers with exact confinement of quarks (permitted by the incoherence of the interior isohilbert and exterior Hilbert spaces), and other intriguing possibilities, such as the regaining of convergent perturbative series for strong interactions (which is possible whenver $|T| << 1$) [68];

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(11) Numerical representation of Arp's measures on quasars redshift as being due to the decrease of speed of light in chromospheres and its isominkowskian geometrization [37].

(12) Numerical representation of the joint redshift and blushifts of pairs of quasars, particularly when proved via gamma spectroscopy to be physically connected to the associated galaxies, and prediction of a measureable isominkowskian redshift for sunlight at sunset [67].

(13) Application to local realism via the proof that Bell's inequality,von Neumann's theorem and all that are inapplicable (rather than "violated") under isotopies (evidently because of the nounitary Structure of the lifting), thus permitting an isotopic completion of quantum mechanics much along the celebrated *E-P-R* argument [65];

(14) Application to q-deformations, discrete time theories and other ongoing studies via their axiomatization into a form invariant under their own time evolution and which coincide with the conventional quantum mechanical axiomatization at the abstract level [33]; and other applications (see monographs [61] and [62];

(14) Novel possibilities in theoretical biology, such as a quantitative representation of the growth of sea shells which, according to computer simulations, crack during their growth is subjected to the conventional Minkowskian geometry, while admit a normal growth under the covering isominkowskian geometry of Class III (the letter one being needed to represent bifurcations which require inversions of time [60].

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LAURICELLA'S FUNCTIONS AND THEIR TRANSFORMATIONS. I

By

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ABSTRACT

In this paper we shall obtain a transformation for Lauricella's function with the help of difference operators.

1. INTRODUCTION.

It is well known that difference operators Δ and E play a very impotant role in the field of special functions. Agarwal [1, 2, 3] and Gupta and Agarwal [5, 6, 7] have successfully applied these operators to obtain various tranformations and also evaluated several integrals in a very simple manner. In this paper we shall obtain the transformation [2, p. 116]

$$F_A[a; b_1, b_2, ..., b_n; c_1, c_2, ..., c_n; x_1, x_2, ..., x_n]$$

$$= (1 - x_1 - ... - x_k)^{-a} F_A[a; c_1 - b_1, c_2 - b_2, ..., c_k - b_k, b_{k+1},$$

$$b_{k+2}, ..., b_n; c_1, ..., c_n; \frac{-x_1}{1 - x_1 - ... - x_k}, ...,$$

$$\left.\frac{-x_k}{1 - x_1 - ... - x_k}, \frac{x_{k+1}}{l - x_1 - ... - x_k}, ..., \frac{x_n}{1 - x_1 - ... - x_k}\right] \quad (*)$$

for Lauricella's functions with the help of difference operators.

2. FIRST STEP OF PROOF

We know that

$$F_2(a; b_1, b_2; c_1, c_2; x_1, x_2]$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a)_{m_1+m_2} (b_1)_{m_1} (b_2)_{m_2} x_1^{m_1} x_2^{m_2}}{(c_1)_{m_1} (c_2)_{m_2} m_1!\, m_2!}$$

$$= \frac{\Gamma(c_1)\,\Gamma(c_2)}{\Gamma(b_1)\,\Gamma(b_2)} (1 - x_1 E_1 - x_2 E_2)^{-a} \frac{\Gamma(b_1)\,\Gamma(b_2)}{\Gamma(c_1)\,\Gamma(c_2)},$$

where $E_1^n f(\alpha_1) = f(\alpha_1 + n)$ and E_i operates on b_i and c_i only such that

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$$\frac{\Gamma(c_i)}{\Gamma(b_i)} E_i^n \left(\frac{\Gamma(b_i)}{\Gamma(c_i)} \right) = \frac{(b_i)_n}{(c_i)_n} \quad \ldots (**)$$

This suggests a generalization of this method i.e., consider

$$A = \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \right] \cdot (1 - x_1 E_1 - \ldots - x_n E_n)^{-a} \left[\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)} \right] \quad \ldots (1)$$

Now we shall show with the help of following Lemmas that this general form reduces to Lauricella's function F_A.

Lemma 1. It is well known that (multinomial Theorem)

$$(x_1 + \ldots + x_n)^k = \sum_{\Sigma m_i = k} \frac{k!\, x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!}.$$

Lemma 2 (Srivastava. [8, p. 4]).

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} (x_1 + \ldots + x_n)^k$$

$$= \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n}}{m_1! \ldots m_n!} x_1^{m_1} \ldots x_n^{m_n}.$$

It can be easily proved either by induction method [8, p. 4] or with the help of Lemma 1.

Now consider

$$A = \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \right]$$

$$\cdot (1 - x_1 E_1 - \ldots - x_n E_n)^{-a} \left[\frac{\Gamma(b_1, \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)} \right],$$

$$= \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \right] \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \cdot (x_1 E_1 + \ldots + x_n E_n)^k \left[\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)} \right].$$

In view of Lemma 1 and Lemma 2 we have,

$$A = \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \right] \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n}}{m_1! \ldots m_n!}$$

$$\cdot (x_1 E_1)^{m_1} \ldots (x_n E_n)^{m_n} \left[\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)} \right]$$

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$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}}{m_1! \ldots m_n!} \frac{(b_1)_{m_1} \ldots (b_n)_{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \cdot x_1^{m_1} \ldots x_n^{m_n}.$$

$$= F_A [a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n], \quad \ldots (2)$$

where, for convergence,

$$|x_1| + x_2| + \ldots + |x_n| < 1.$$

It has now been shown that A equals the lhs of (*).

3. SECOND STEP

To prove that A equals also the rhs of (*) we set $E_i = 1 + \Delta_i$. It can be shown that

$$\frac{\Gamma(c_i)}{\Gamma(b_i)} \Delta_i^m \left(\frac{\Gamma(b_i)}{\Gamma(c_i)} \right) = \frac{(-1)^m (c_i - b_i)_m}{(c_i)_m}. \quad \ldots (***)$$

Thus,

$$A = \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \right] (1 - x_1 E_1 - \ldots - x_n E_n)^{-a} \left(\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)} \right)$$

$$= \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \right] (1 - x_1 - \ldots - x_k)^{-a}$$

$$\cdot \left[1 - \frac{x_1 \Delta_1 + \ldots + x_k \Delta_k + x_{k+1} E_{k+1} + \ldots + x_n E_n}{1 - x_1 - \ldots x_k} \right]^{-a} \left(\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)} \right)$$

Again using Lemma 1 and Lemma 2 we have,

$$A = \left[\frac{\Gamma(c_1) \ldots \Gamma(c_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \right] (1 - x_1 - \ldots x_k)^{-a}$$

$$\cdot \sum_{m_1=0}^{\infty} \ldots \cdot \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}}{m_1! \ldots m_n!} \left(\frac{x_1 \Delta_1}{1 - x_1 - \ldots - x_k} \right)^{m_1}$$

$$\ldots \left(\frac{x_k \Delta_k}{1 - x_1 \ldots - x_k} \right)^{m_k} \left(\frac{x_{k+1} E_{k+1}}{1 - x_1 - \ldots - x_k} \right)^{m_{k+1}} \ldots$$

$$\left(\frac{x_n E_n}{1 - x_n - \ldots - x_k} \right)^{m_n} \left(\frac{\Gamma(b_1) \ldots \Gamma(b_n)}{\Gamma(c_1) \ldots \Gamma(c_n)} \right),$$

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Hence, from (1), (2) and (***), we see that A also equals the rhs of (*).

This completes the proof.

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LEFT ALTERNATIVE RINGS WITH AN IDEMPOTENT

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ABSTRACT

Let R be a left alternative ring that satisfies $(x, y, z) + (y, z, x) + (z, x, y) = 0$. An idempotent element in R belongs to the nucleus of R. Let R be a prime ring with characteristic prime to 6. If R has an idempotent $e \neq 0, 1$ then R is associative. We give an example of a ring under consideration with an idempotent $e \neq 0, 1$.

1. INTRODUCTION

A non-associative ring R is called left alternative if

$$(x, y, z) + (y, x, z) = 0 \qquad \ldots (1)$$

for all x, y, z in R where the associator $(x, y, z) = (xy)z - x(yz)$. That is,

$$(x, x, z) = 0 \qquad \ldots (i)$$

for all x, z in R.

Kleinfed and Smith [1] have shown that a prime left alternative ring with commutators in the left nucleus and characteristic $\neq 2, 3$ is associative. Rich (2) has considered rings with idempotents in their nuclei.

2. PRELIMINARIES

In any arbitrary ring we have

$f(w, x, y, z) = (wx, y, z) - (w, xy, z) + (2, x, yz) - w(x, y, z) - (w, x, y)z = 0$

Then using (1) in $f(x, y, y, z) - f(y, y, x, z) + f(y, x, y, z) = 0$

we get $2(xy, y, z) - 2y(x, y, z) = 0$. Assuming that characteristic $\neq 2$,, we have

$$(xy, y, z) = y(x, y, z) \qquad \ldots (2)$$

Linearizing (2) we get

$$g(x, w, y, z) = (xw, y, z) + (xy, w, z) - w(x, y, z) - y(x, w, z) = 0.$$

Using (1) and (2) in $f(x, y, y, z) = 0$, we get

$$(x^2, y, z) = (x, xy + yx, z).$$

3. MAIN SECTION

Unless otherwise stated R will be a left-alternative ring that satisfies

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$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \quad \ldots (4)$$

for all x, y, z in R.

Lemma 1. Let A be a non-zero ideal of R. Then $B = \{x \in R : xA = Ax = (0)\}$ is an ideal of R.

Proof. Let $a \in A, x \in B$ and $y \in R$. Then

$$0 = (y, a, x) + (a, y\, x) = (ya)x - y(ax) + (ay)x - a(yx).$$

But $a \in A$ implies that $ya, ay \in A$. Therefore, we have $a(yx) = 0$.

So $(a, y, x) = (ay)x - a(yx) = 0$. Now from (4)

$$0 = (y, x, a) + (x, a, y) + (a, y, x) = (y, x, a) + (x, a, y)$$
$$= (yx)a - y(xa) + (xa)y - x(ay) = (yx)a.$$

Therefore $yx \in B$. Interchanging the role of x and y in above we get $xy \in B$. Hence B is an ideal of R.

Now we assume that $((e, x), e, e) = 0$ where e is an idempotent of R and $(e, x) = ex - xe$.

Theorem 2. Let R be a ring of characteristic prime to 6. If R has an idempotent $e \neq 0, 1$ then R has the desired Peirce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$ where x belongs to R_{ij} if and only $ex = ix$ and $xe = jx$ for $i, j = 0, 1$ the sum of the sub-mdodules is direct.

Proof. It suffices to show that $(e, e, x) = (e, x, e) = (x, e, e) = 0$ for all x in R. Let $x \in R$. Then

$$(e, x, e) = (e^2, x, e) = (e, ex + xe, e) \quad \text{(by (3))}$$
$$= (e, ex, e) + (e, xe, e)$$

Now $0 = ((e, x), e, e) = (e, (e, x), e)$ (by (1))

This implies that $(e, ex, e) = (e, xe, e)$. Therefore, $(e, x, e) = 2(e, ex, e)$. Therefore,

$(e, x, e) = 2(e, ex, e)$. That is,

$$(e, x, e) - 2(e, ex, e) = 0 \quad \ldots (5)$$

for all x in R. In particular,

$$(e, ex, e) - 2(e, e(ex), e) = 0.$$

But since R is left-alternative $(e, e, x) = 0$. Therefore, $e(ex) = (ee)x = ex$.

Thus, $(e, ex, e) = 0$.

Substituting in (5), we get $(e, x, e) = 0$.

Now $(e, x, e) + (x, e, e) + (e, e, x) = 0$ implies that $(x, e, e) = 0$. Therefore $(e, x\, e) = (x, e\, e) = (e, e, x) = 0$.

Hence R has the desired Peirce decomposition.

Lemma 3. Suppose R has a Peirce decomposition with respect to e. Then

(*i*) $R_{ii} R_{ii} \subseteq R_{ii}$

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(*ii*) $R_{ij} R_{kl} \subseteq \delta_{jk} R_{il}$ for all $i, j, k, l = 0, 1$.

Proof. Let $x_{11}, y_{11} \in R_{11}$. Then $e(x_{11} y_{11}) = -(e, x_{11}, y_{11}) + ex_{11}) y_{11}$

$= (x_{11}, e, y_{11}) + x_{11} y_{11} = (x_{11} e) y_{11} - x_{11}(e y_{11}) + x_{11} y_{11}$

$= x_{11} y_{11} - x_{11} y_{11} + x_{11} y_{11} = x_{11} y_{11}$. And

$$(x_{11} y_{11}) e = x_{11}, y_{11}, e) + x_{11}(y_{11} e)$$

$$= -(y_{11}, e, x_{11}) - (e, x_{11}, y_{11}) + x_{11} y_{11} \qquad \text{(by (4))}$$

$$= -(y_{11}, e) x_{11} + y_{11}(ex_{11}) + (x_{11}, e, y_{11}) x_{11} y_{11}$$

$$= -y_{11} x_{11} + y_{11} x_{11} + (x_{11} e) y_{11} - x_{11}(ey_{11}) + x_{11} y_{11}$$

$$= -y_{11} x_{11} + y_{11} x_{11} + x_{11} y_{11} - x_{11} y_{11} + x_{11} y_{11} = x_{11} y_{11}.$$

These imply that $x_{11} y_{11} \in R_{11}$. Therefore $R_{11} R_{11} \in R_{11}$. Similarly other results can also be proved.

Nucleus : For an arbitrary non-associative ring R, the nucleus N is defined by

$$N = \{x \in R : (x, y, z) = (y, z, x) = (y, x, z) = \text{ for all } y, z \in R\}.$$

Lemma 4. Suppose R has an idempotent $e \neq 0, 1$. Then e belongs to nucleus N or R.

Proof. We shall show that $(x, e, y) = 0$ for all x, y in R. This will imply $(e, x, y) = 0$ by (1) and by (4) we get $(x, y, e) = 0$. Let $x, y \in R$ Then $R = R_{11} + R_{10} + R_{01} + R_{00}$ implies that $x = x_{11} + x_{10} + x_{01} + x_{00}$ and $y = y_{11} + y_{10} + y_{01} + y_{00}$ where $x_{ij}, y_{ij} \in R_{ij}$, $i, j = 0, 1$.

Now $(x, e, y) = (xe)y - x(ey) = (x_{11} + x_{01})y - x(y_{11} + y_{10})$

$= x_{11} y_{11} + x_{11} y_{10} + x_{01} y_{11} + x_{01} y_{10} - x_{11} y_{11} - x_{01} y_{11} - x_{11} y_{10} - x_{01} y_{10}$

$= 0 (\because R_{ij} R_{kl} = (0)$, if $j \neq k$, $(i, j) \neq (k, l)$ and either $i \neq j$ or $k \neq l$). Hence $e \in N$.

Lemma 5. Suppose y belongs to the nucleus N of R. Then $(y, z) \in N$ for all z in R.

Proof. Let $x, w, z \in R$ and $y \in N$. Then

$$g(x, w, y, z) = (xw, y, z) + (xy, w, z) - w(x, y, z) - y(x, w, z) = 0.$$

implies that $(xy, w, z) = y(x, w, z) = 0$. Changing x to z, w to x and z to w in the above we get

$$(zy, x, w) = y(z, x, w) \qquad \ldots (6)$$

Now $f(y, z, x, w) = (yz, x, w) - (y, zx, w) + (y, z, xw) - y(z, x, w) - (y, z, x) = 0$. Since $y \in N$, we have

$$(yz, x, w) = (z, x, w) \qquad \ldots (7)$$

From (6) and (7) we get $((y, z) x, w) = 0$. By (1) $(x, (y, z), w) = 0$ and by (4) $(x, w, (y, z)) = 0$ for all x, w in R. Hence $(y, z) \in N$.

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Definition. A ring R is said to be *prime* if for any two ideals A and B of R, $AB = (0)$ implies $A = (0)$ or $B = (0)$.

Lemma 6. Let R be an arbitrary non-associative prime ring. Then R can contain no non-zero nuclear ideals (ideals in nucleus).

Proof. Let A be an ideal in the nucleus N of R. Let $x, y, z, w \in R$ and $a \in A$. Then

$$f(a, x, y, z) = (ax, y, z) - (a, xy, z) + (a, x, yz) - a(x, y, z) - (a, x, y)z = 0.$$

Since ax , $a \in A \subseteq N$, we get $a(x, y, z) = 0$.

Because $a \in A \subseteq N$, $a((x, y, z)w) = (a(x, y, z))w = 0$.

This implies that $A((R, R, R) + (R, R, R)R) = (0)$. But $(R, R,R) + (R, R, R)\,R$ is an ideal of R and R is prime. Either $A = (0)$ *or* $(R, R, R) + (R, R, R)R = (0)$. But R is non-associative. THerefore $= (0)$. This proves the lemma.

Lemma 7. Suppose N is the nucleus of R. Then

(*i*) $N(R, R, R) = (NR, R, R)$.

(*ii*) $(R, R, R)N = (R, R, RN)$.

(*iii*) $N(R, R, R) = (R, NR, R)$.

(*iv*) $[N, (R, R, R)] = (0)$.

Proof. Let $n \in N$ and $x, y, z \in R$. Then

(*i*) $f(n, x, y, z) = (nx, y, z) - (n, xy, z) + (n\ , x, yz) - n(x, y, z)$
$- (n\ , x, y)z = 0$

implies that $(nx, y, z) = n(x, y, z)$. Therefore $(NR, R, R) = N(R, R, R)$.

(*ii*) $f(x, y, z, n) = (xy, z, n) - (x, yz, n) + (x, y, zn) - x(y, z, n)$
$- (x, y, z)n = 0$

implies that $(x, y, zn) = (x, y, z)n$. Therefore $(R, R, RN) = (R, R, R)N$.

(*iii*) $(x, ny, z) = (ny, x, z) = - n(y, x, z) = n(x, y, z)$

(*iv*) $f(x, n, y, z) = (xn, y, z) - (x, ny, z) + (x, n\ yz) - x(n, y, z)$
$- (x, n, y) = 0$

Therefore $(R, NR, R) = N(R, R, R)$.

implies that

$$(x, ny, z) = (xn, y, z) \qquad \ldots (8)$$

By (*ii*) $(x, y, z)n = (x, y, zn)$

$= - (y, zn, x) - (zn, x, y)$ (by (4))

$= (zn, y, x) - (zn, x, y)$ (by (1))

$= (z, ny, x) - (z, nx, y)$ (by (8))

$= - (ny, z, x) + (nx, z, y)$ (by (1))

$= - n(y, z, x) + n(x, z, y)$ (by (*i*))

$= n((z, y, x) + (x, z, y))$ (by (1))

$= - n(y, x, z)$ (by (4))

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$= n(x, y, z)$ (by (1))

Therefore $[n, (x, y, z)] = 0$. Hence $[N, (R, R, R)] = (0)$.

Corollary 8. $[R, N] \subseteq N$.

Proof. Let $x, y, z \in R$ and $n \in N$. Then using (i), (iii) and (8), $(nx, y, z) = n(x, y, z) = (x, ny, z) = (xn, y, z)$ Therefore, $((x, n), y, z) = 0$. Using (1), $(y, (x, n), z) = 0$.

Using (4), $(y, z\ (x, n)) = 0$. Hence $(R, N) \subseteq N$.

Lemma 9. Suppose N is the nucleus of R. If I is the ideal generated by the set $[R, N)$, then $I = [R, N] + R[R, N]$.

Proof. Clearly $[R, N] + R[R, N] \subseteq I$. Let $n_1, n_2 \in N$ and $x, y, z, w \in R$. Then $[x, n_1]w = [(x, n_1), w] + w[x, n_1]$ implies that $[x, n_1]\ w \in [R, N] + R[R, N]$ since $(x, n_1) \in (R, N) \subseteq N$. Also, since $[R, N] \subseteq N$, $(y, [z, n_2])w = -y(z, n_2]w) = y[z, n_2)w] + y(w[z, n_2]) \in R[R, N] + R[R, N]$. Therefore $[R, N]$ is a right deal. Because $[R, N] \subseteq N$,

$R(R, N] + R[R, N]) = R[R, N] + R(R[R, N)) = R[R, N] + (RR)\ [R, N]$

$\subseteq R[R, N] + R[R, N] \subseteq [R, N] + R[R, N]$.

Thus $[R, N] + R[R, N]$ is a two sided ideal of R. It contains $[R, N]$ which is contained in I. Hence $I = [R, N] + R[R, N]$.

Lemma 10. In the Peirce decomposition $R_{10}^2 = R_{01}^2 = (0)$.

Proof. Let $x_{10}, y_{10} \in R_{10}$ and $e \in N$. Then $(x_{10}, e, y_{10}) = 0$ implies that $(x_{10}\, e)\, y_{10} = x_{10}(ey_{10})$ or $x_{10}y_{10} = 0$. Similarly considering $(x_{01}, e, y_{01}) = 0$ we can show that $x_{10}\, y_{01} = 0$. Hence $R_{10}^2 = R_{01}^2 = (0)$.

Lemma 11. Suppose R has an idempotent $e \neq 0, 1$. Then the set $B = R_{10}\, R_{01} + R_{10} + R_{01} + R_{01}\, R_{10}$ is an ideal of R contained in the nucleus N of R.

Proof By Lemma 4, $e \in N$. By Lemma 5, $(e, x_{01}) \in N$ for some $x_{01} \in R_{01}$. This impies that $x_{01} \in N$ or $R_{01} \subseteq N$. Again $(e, x_{10}) \in N$ for some $x_{10} \in R_{10}$ implies that $R_{10} \subseteq N$. Since N is a sub- ring of R, we have both $R_{10}\, R_{01}$ and $R_{01}\, R_{10}$ contained in N. Hence $B \subseteq N$. Now using Lemma 2 and the fact that $R_{10}, R_{01} \subseteq N$, we have.

$$
\begin{aligned}
RB &= (R_{10} + R_{01} + R_{00} + R_{11})\,(R_{10} + R_{10}\, R_{01} + R_{01}\, R_{10} + R_{01}) \\
&\subseteq R_{10}^2 + R_{10}\,(R_{10}\, R_{01}) + R_{10}\,(R_{01}\, R_{10}) + R_{10}\, R_{01} + R_{01}\, R_{10} + R_{01}\,(R_{10}\, R_{01}) \\
&\quad + R_{01}\,(R_{01}\, R_{10}) + R_{01}^2 + R_{00}\, R_{10} + R_{00}\,(R_{10}\, R_{01}) + R_{00}\,(R_{01}\, R_{10}) + R_{00}\, R_{01} \\
&\quad + R_{11}\, R_{10} + R_{11}\,(R_{10}\, R_{01}) + R_{11}\,(R_{01}\, R_{10}) + R_{11}\, R_{01} \\
&\subseteq O + R_{10}\, R_{11} + R_{10}\, R_{00} + R_{10}\, R_{01} + R_{01}\, R_{10} + R_{01}\, R_{11} + R_{01}\, R_{00} \\
&\quad + O + O + R_{00}\, R_{11} + (R_{00}\, R_{01})\, R_{10} + R_{01}\, R_{10} + (R_{11}\, R_{10})\, R_{01} + R_{11}\, R_{00} + O
\end{aligned}
$$

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$$\subseteq O + O + R_{10} + R_{10}R_{01} + R_{01}R_{10} + R_{01} + O + O + O + O + R_{01}R_{10}$$
$$+ R_{01} + R_{10} + R_{10}R_{01} + O + O \subseteq B$$

Similarly we can show that $BR \subseteq B$. Hence B is an ideal of R contained in N.

Theorem 12. Suppose R is a prime ring of characteristic prime to 6 has an idempotent $e \neq 0, 1$. Then R is associative.

Proof. Suppose R is not associative. Then by lemma 6, R can contain no non-zero nuclear ideal. By lemma 11, $B = (0)$. This implies that both R_{10} and R_{01} are (0) By theorem 2, $R = R_{00} + R_{11}$. Now, by lemma 3, $R_{00}(R_{00}(R_{00} + R_{11}) \subseteq R_{00}R_{00} + R_{00}R_{11} \subseteq R_{00}$ and $(R_{00} + R_{11})R_{00} \subseteq R_{00}R_{00} + R_{11}R_{00} \subseteq R_{00}$. Therefore, R_{00} is an ideal of R. Similarly R_{11} is an ideal of R. Let $x \in R_{00}$ and $x \in R_{11}$ Then $(x, e, y) = 0$ implies $(xe)y = x(ey)$ which implies $xy = 0$. Therefore $R_{00}R_{11} = (0)$ But R is prime. Either $R_{00} =$ or $R_{11} = (0)$. But $e \neq 0 \in R_{11}$. Therefore, $R_{00} = (0)$. Then $R = R_{11}$. So $ex = xe = x$ for all x in R. Thus $e = 1$. Hence R is associative.

Example 13. Let R be a ring defined by the following multiplication table together with all finite sums of e, a, b

	e	a	b
e	e	0	$-b$
a	b	0	0
b	0	0	0

Note that $(e, a, e) = (ea)e - e(ae) = -eb = b$. Therefore, R is non-associative.

To prove $(x, y, z) + (y, x, z) = 0 \ \forall \ x, y, z \in R$, it suffices to check that

$(e, a, e) + (a, e, e) = 0$.

Now $(a, e, e) = (ae)e = a(ee) = be - ae = 0 - b = -b$

$\therefore \ (e, a, e) + (a, e, e) = b - b = 0$.

Also $(e, e, a) = (ee)a - e(ea) = ea = 0$.

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ON A CLASS OF ENTIRE FUNCTIONS REPRESENTED BY TAYLOR DIRICHLET SERIES

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ABSTRACT

In this paper, we have studied certain classes of Taylor-Dirichlet series. The standard Taylor series and Dirichlet series turn out to be the particular cases of this series. Various functional analytic structures have been provided to these classes. Topological zero divisors, invertible elements and continuous linear functionals have been investigated for these classes. An interesting method of construction of total sets has also been formulated.

1. Introduction

In (3) Rishishwar has studied the series of type

$$(1.1) \qquad f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n \psi(z)}$$

where,

$$\lim_{n \to \infty} \sup . \frac{n}{\lambda_n} = D < \infty, \; 0 < \lambda_1 < \lambda_2 < \ldots \; (\lambda_n \to \infty \text{ as } n \to \infty).$$

$\{a_n\}$ $(n = 1, 2, 3, \ldots)$ is a sequence of complex numbers $\psi(z)$ is a function of complex variable z.

The standard Taylor series and Dirichlet series turn out to be the particular cases of the series (1.1). Hence, in future we shall call the series (1.1) as Taylor Dirichlet series (TDS-in short).

Suppose that

$$\psi(z) = \phi(-z) + i\,\theta(z)$$

where Φ are real valued functions of the complex variable z and that ψ is invertible and continuous. TDS converges in the half plane (cf. [3]) Re. $\psi(z) = \varphi(z) \leq R$. Therefore TDS represents an entire function if and only if

$$(1.2) \qquad \varphi^{-1}\left[\lim_{n \to \infty} \inf. \frac{\log |a_n|}{-\lambda_n}\right] = \infty$$

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Let E stands for the family of all entire Taylor Dirichlet series (ETDS - in short). The function $f(z), g(z)$ and $h(z)$ are defined as

$$f(z) = \sum_{n=1}^{\infty} a_n^{\lambda_n \psi(z)}$$

$$g(z) = \sum_{n=1}^{\infty} b_n e^{\lambda_n \psi(z)}$$

and

$$h(z) = \sum_{n=1}^{\infty} c_n e^{\lambda_n \psi(z)}.$$

Throughout this paper, summation extends from 1 to ∞, unless limits are specified and f, g, h stands for $f(z), g(z)$ and $h(z)$ respectively.

2. Let F be the subset of E defined by

$$F = \{f(z) : f(z) = \Sigma a_n e^{\lambda_n \psi(z)}, \sup_{n \geq 1} . e^{\lambda_n \varphi(n)} |a_n| < \infty\}$$

It can easily be seen that every element of F represents a function which converges in the whole complex plane. With pointwise addition and scalar multiplication, F becomes a linear space. The norm in F is defined as follows.

$$\|f\| = \sup_{n \geq 1} e^{\lambda_n \varphi(n)} |a_n|, f(z) = \Sigma a_n e^{\lambda_n \psi(z)} \in F.$$

It can easily be varified that F together with this norm defined on it, is a normed linear space. Multiplication in F is defined as

$$f.g = \Sigma e^{\lambda_n \varphi(n)} a_n b_n e^{\lambda_n \psi(z)}, \text{ for every } f, g, \in F.$$

For the definition of terms used we refer [1].

Theorem 1. F is a commutative B^*-algebra with identity.

Algebra- We just note that

$$\|f.g\| = \sup_{n \geq 1} e^{\lambda_n \varphi(n)} | e^{\lambda_n \varphi(n)} a_n b_n |$$

$$\leq \sup_{n \geq 1} e^{\lambda_n \varphi(n)} |a_n| \sup_{n \geq 1} . e^{\lambda_n \varphi(n)} |b_n|$$

$$= \|f\| . \|g\|.$$

Identity. The element $e = \Sigma e_n e^{\lambda_n \psi(z)} \in F$, where $e_n = e^{-\lambda_n \varphi(n)}$, serves as identity element in F.

Completeness. Let $\{f_p\}$ be a Cauchy sequence in F, where

$$f_p(z) = \Sigma a_{pn} e^{\lambda_n \psi(z)}$$

Given $\epsilon > 0$ we can find $p_0 \geq 1$ such that

$$\|f_p - f_q\| < \varepsilon, \text{ for every } p, q \geq p_0$$

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i.e.

$$\sup_{n \geq 1} e^{\lambda_n \varphi(n)} \, |a_{pn} - a_{qn}| < \varepsilon, \text{ for every, } p\, q \geq p_0.$$

This implies that $\{a_{pn}\}$ is a Cauchy sequence in C for every n and hence, owing to the completeness of C converges to a complex number say, a_n.

Thus

$$f_p \to f, \text{ where } f(z) = \Sigma a_n \, e^{\lambda_n \psi(z)}.$$

Moreover $f(z)$ is a member of F. Since

$$e^{\lambda_n \varphi(n)} \, |a_n| \leq e^{\lambda_n \varphi(n)} \, |a_{pn} - a_n| + e^{\lambda_n \varphi(n)} |\, a_{pn}|$$

Involution. The operation * defined of F by

$$*(f) = f^*(z) = \Sigma \, \overline{a}_n \, e^{\lambda_n \psi(z)}$$

is an involution mapping. We note that

$$\| f.f^* \| = \sup_{n \geq 1} . e^{\lambda_n \varphi(n)} \, |e^{\lambda_n \varphi(n)} a_n . \overline{a}_n|$$

$$= \sup_{n \geq 1} \{e^{\lambda_n \varphi(n)} \, |a_n|\}^2$$

$$= \|f\|^2.$$

Rest of the proof is straight forward.

Corollary. $(F, \|.\|)$ is a Gelfand Algebra.

Proof. Since $\|e\| = \sup_{n \geq 1} e^{\lambda_n \varphi(n)} \, | e^{-\lambda_n \varphi(n)}| = 1$

Theorem 2. A function $f(z) = \Sigma \, a_n \, e^{\lambda_n \psi(z)}$ in F is a topological zero divisor if and only if

(2.1) $$a_n = 0 \, (e^{-\lambda_n \varphi(n)})$$

Proof. Consider the sequence $\{g_k\}$ in F, were

$$g_k(z) = e^{\lambda_k [\psi(z) - \varphi(k)]}, \, k \geq 1$$

Note that for every $k \geq 1$. $g_k \in F$ and $\|g_k\| = 1$. Moreover,

$$f \cdot g_k = g_k \cdot f = e^{\lambda_k \varphi(k)} \, a_k \, . \, e^{-\lambda_k \varphi(k)} \, . \, e^{\lambda_k \psi(z)}$$

$$= a_k \, . \, e^{\lambda_k \psi(z)}.$$

Hence owing to (2.1)

(2.2) $$\|f \cdot g_k\| = \|g_k \cdot f\| \to 0 \text{ as } k \to \infty$$

Thus $f(z)$ is a topological zero divisor in F. Let on the other hand $f(z) = \Sigma a_n \, e^{\lambda_n \psi(z)}$ be a topological zero divisor in F. There exists

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therefore a sequence $\{g_k\}$ of elements in F with unit norm such that (2.2) holds.

Let of possible, $\lim_{n \to \infty} e^{\lambda_n \varphi(z)} |a_n| = l \neq 0$. Thus for an arbittary $\gamma > 0$, there exists n_0 such that

$$e^{\lambda \varphi(n)} |a_n| > l - \gamma, \quad \text{for } n > n_0. \tag{2.3}$$

By hypothesis, for every $k > 1$

$$\sup_{n \geq 1} e^{\lambda_n \varphi(n)} |b_{kn}| = 1.$$

Given $\varepsilon > 0$, we can find for every k; an integer N_k and a subsequence $\{n_i\}$ of indices such that

$$e^{\lambda_n \varphi(n)} |b_{k \cdot n_1}| > 1 - \varepsilon \quad \text{for} \quad n_i > N_k. \tag{2.4}$$

(2.3) and (2.4) together imply that

$$\| f.g_k \| > C > 0.$$

Contradicting (2.2). Hence the theorem.

Note that F is not a division algebra. For instance, the element

$$p(z) = \Sigma \lambda_n^{-1} e^{-\lambda_n \varphi(n)} e^{\lambda_n \psi(z)}$$

though belongs to F, does not possess inverse in F. For, if possible, $q(z) = \Sigma d_n e^{\lambda_n \psi(z)}$ be its inverse. Hence we must have

$$p \cdot q = e$$

i.e.

$$e^{\lambda_n \varphi(n)} (e^{-\lambda_n \varphi(n)} \lambda_n^{-1} d_n) = e^{-\lambda_n \varphi(n)}$$

This implies

$$d_n = \lambda_n e^{-\lambda_n \varphi(n)}.$$

It can easily be varified that the so defined $q(z) = \Sigma d_n e^{\lambda_n \psi(z)}$ is not a member of F. In this connection we propose.

Theorem-3. The function $f(z) = \Sigma a_n e^{\lambda_n \psi(z)}$ is invertible in F if and only if $\{|e^{\lambda_n \psi(n)} a_n|^{-1}\}$ is a bounded sequence.

Proof. Let $f(z)$ be invertible and let $g(z)$ be its inverse. By hypothesis we have

$$e^{\lambda_n \psi(n)} a_n b_n \equiv e^{-\lambda_n \varphi(n)}$$

$$e^{\lambda_n \phi(n)} |b_n| = |e^{\lambda_n \phi(n)} a_n|^{-1}$$

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Since $g(z) = \Sigma b_n e^{\lambda_n \varphi(z)} \in F$. We see that $\{|e^{\lambda_n \varphi(z)} a_n|^{-1}\}$ is a bounded sequence

On the other hand, if $\{|e^{\lambda_n \varphi(z)} a_n|^{-1}\}$ bounded sequence, define $g(z)$ such that

$$g(z) = \Sigma e^{-2\lambda_n \phi(n)} a_n^{-1} e^{\lambda_n \psi(z)}$$

Obviously, $g(z) \in F$. Moreover,

$$f.g = \Sigma e^{\lambda_n \varphi(n)} a_n e^{-2\lambda_n \varphi(n)} a_n^{-1} e^{\lambda_n \psi(z)}$$
$$= \Sigma e^{-\lambda_n \varphi(n)} e^{\lambda_n \varphi(z)}$$
$$= e(z).$$

Thus,

$$f.g = e.$$

In this section we consider a class G of ETDS such that

$$G = \{f : f(z) = \Sigma a_n e^{n\varphi(z)}, \Sigma e^{\lambda_n \varphi(z)} \mid a_n| < \infty\}$$

The norm in G is defined as

$$\|f\| = \Sigma e^{\lambda_n \varphi(z)} \mid a_n|.$$

With similar binary composition as in F and the norm defined in G, G becomes a normed linear space.

Theorem 4. G is a Banach* - Algebra.

Proof. We only prove the completeness.

Let $\{f_p\}$ be a Cauchy sequence in G. Given $\varepsilon > 0$, there exists some $p_0 \geq 1$, such that

$$\|f_p - f_q\| < \varepsilon, \text{ got } p, q, \geq p_0$$

$$e^{\lambda_n \varphi(z)} |a_{pn} - a_{qn}| < \varepsilon, \text{ for } p, q \geq p_0. \tag{3.1}$$

This implies that $\{a_{pn}\}$ forms a Cauchy sequence in C for every n and hence, owing to the completeness of C, converges to a complex number, say, a_n. In (3.1), let $q \to \infty$. We get

$$e^{\lambda_n \varphi(n)} \mid a_{pn} - a_n| < \varepsilon, p \geq p_0$$

$f_p \to f$, where $f(z) = \Sigma a_n e^{\lambda_n \varphi(z)} \in G$, since

$$\Sigma e^{\lambda_n \varphi(z)} |a_n| \leq e^{\lambda_n \varphi(n)} |a_{pn} - a_n| + \Sigma e^{\lambda_n \varphi(n)} |a_{pn}|.$$

It can now be verified that G is a Banach algebra.

The proof of the theorem is complete when we define involution mapping.

$$* : G \to G$$

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as

$$* (f) = f * (z) = \Sigma\, \overline{a}_n\, e^{\lambda_n \psi (z)}$$

Note that G does not becomes a B*-algebra.

Theorem 5. Every continuous linear functional f^* on G is of the form

(3.2) $$f * (f) = \Sigma\, e^{\lambda_n \varphi(z)}\, a_n \,.\, d_n \text{, where}$$

(3.3) $\{d_n\}$ is a bounded sequence.

Conversely if (3.3) holds, then (3.2) defines a continuous linear functional on G.

Proof. We shall denote that dual space of G by G^*. Let $f^* \in G^*$. Define

$$f_n = e^{\lambda_n \{\psi (z) - \varphi (n)\}} \text{ and } f^N = \sum_{n=1}^{N} a_n\, e^{\lambda_n \psi (z)}$$

obviously, $f^N \to f$ as $N \to \infty$. Let

$$f * (f_n) = d_n.$$

Then

$$f * (f) = f * (\lim_{N \to \infty} f^N)$$

$$= f * (\lim_{N \to \infty} \sum_{n=1}^{N} e^{\lambda_n \varphi(z)}\, a_n f_n)$$

$$= \sum_{n=1}^{\infty} e^{\lambda_n \varphi (n)}\, a_n \,.\, d_n$$

Moreover,

$$| d_n | = | f * (f_n) | \leq M. \; \| f \| = M.$$

Conversely, let (3.3) holds. The functional defined by (3.2) is evidently well defined and linear. Further we note.

(3.4) $$\| f* \| \leq \Sigma e^{\lambda_n \varphi (n)} \, | a_n d_n | \leq M \; \| f \|, \qquad \text{by (3.3).}$$

This characterization helps us in formulating an alternative expression for the norm is G^*. We know from (2) that G is a Banach space with the same operations as in G and the norm defined as

$$\| G* \| = \sup_{\| f \| \leq 1} | f * (f) | / \| f \|.$$

We assert that

$$\| f * \| = \sup_{n \geq 1} | d_n |.$$

By (3.4) we infer that

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$$||f^*|| = \sup_{|f| \le 1} |f^*(f)| / \|f\| \le \sup_{n \ge 1} |d_n|.$$

on the other hand

$$|d_n| = |f^*(f_n)| \le \|f^*\| . \|f_n\| = \|f^*\|$$

Hence the assertion.

Theorem 6. Let $f(z) = \Sigma a_n e^{\lambda_n \varphi(z)} \in G$, $a_n \neq 0$, for every, $n \ge 1$. Let B be the set of complex numbers having at least one finite limit point. Define

$$f_\alpha(z) = \Sigma a_n e^{\lambda_n \{\psi(z) + \psi(\alpha - \varphi(n))\}}.$$

Then the set

$$S_f = (f_\alpha : \alpha \in B)$$

is a total set in G

Proof. Note first that $f_\alpha \in G$, for every $\alpha \in \beta$.

Since

$$f_\alpha(z) = \Sigma e^{\lambda_n \{\psi(z) - \varphi(n)\}} a_n . e^{\lambda_n \psi(z)}, \text{ and}$$

$$\Sigma e^{\lambda_n \varphi(n)} \left| a_n e^{\lambda_n \{\psi(\alpha) - \varphi(n)\}} \right| = \Sigma |a_n| e^{\lambda_n \varphi(\alpha)}$$

where $\varphi(\alpha) = \text{Re } \psi(\alpha)$,

which must converges for every $\alpha \in C$, since $f(z)$ is a function which converges absolutely in the whole complex plane.

Let $f^* \in G^*$ be such that $f^*(s_f) = 0$ *i.e.*

$$f^*(f_\alpha) = 0, \text{ for every } \alpha \in B$$

implies that $\Sigma e^{\lambda_n \psi(n)} a_n e^{\lambda_n \{\psi(\alpha) - \varphi(n)\}} d_n = 0$ for every $\alpha \in B$.

Implies that

(3.5) $$a_n e^{\lambda_n \psi(\alpha)} d_n = 0, \text{ for every } \alpha \in B.$$

Define $h(z) = \Sigma a_n e^{\lambda_n \psi(z)} d_n$. Since (3.5) holds and $f = \Sigma a_n e^{\lambda_n \psi(z)} \in G$, $h = \Sigma a_n e^{\lambda_n \psi(z)} d_n \in G$. But owing to (3.5)

$$h(\alpha) = 0; \text{ for every } \alpha \in B.$$

Since B has a finite limit point, this means that $h \equiv 0$. This however, implies that $a_n d_n = 0$, for every $n \ge 1$, and as a_n is non zero for every n, we get the result.

Theorem 7. Every element of G is a topological zero divisor is G.

Proof. For the definition of topological zero divisor we refer to [1]. Consider the sequence $\{g_k\}$ where

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$$g_k(z) = \Sigma\, e^{-\lambda_k \varphi(k)}\ e^{\lambda_k \psi(z)},\ k = 1, 2, 3 \ldots$$

obviously $g_k \in G$ and $\|g_k\| = 1$, and for every $k \geq 1$.

Also,

$$f \cdot g_k = g_k \cdot f = a_k \cdot e^{\lambda_k \psi(z)}$$

So,

$$\|f.g_k\| = \|g_k \cdot f\| = e^{\lambda_k \varphi(k)}\ |a_k| \to 0 \text{ as } k \to \infty.$$

Hence, the theorem.

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SOME FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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ABSTRACT

In this paper we prove fixed point theorems in a complete metric space which extend the theorem 2 of Khan, Swaleh and Sessa [1] and theorem 2 of H.K. Pathak and Rekha Sharma [2].

Introduction

Let R^+ be the set of non-negative real numbers and N the set of positive integers. Khan, Swaleh and Sessa [1] have established fixed point theorems for self maps of complete metric spaces by altering the distance between the points with the use of a function $\psi : R^+ \to R^+$ satisfying the following properties :

(1) ψ is continuous and increasing in R^+;

(2) $\psi(t) = 0$ if and only if $t = 0$.

We denote the set of above function ψ with φ.

In [1, Th.2] the following theorem was proved :

Theorem 1. Let (x, d) be a complete metric space, T a self map of X, and $\psi : R^+ \to R^+$ an increasing, continuous functions satisfying property (2).

Further more, let a, b, c, be three decreasing functions from $R^+/\{0\}$ into $[0, 1]$ such that $a(t) + 2b(t) + c(t) < 1$ for every $t > 0$. Suppose that T satisfies the following condition :

(A) $\psi(d(Tx, Ty)) \le a(d(x, y))\ \psi(d(x, y)) + b(d(x, y))\ \{\psi(d(x, Tx)) + \psi(d(y, Ty))\}$
$+ c(d(x, y))\ \min\{\psi(d(x, Ty)), \psi(d(y, Tx))\}$

where $x, y \in X$ and $x \neq y$. Then T has a unique fixed point.

In (2, Th. 2) the following theorem was proved :

Theorem 2. Let (X, d) be a complete metric space, T a selfmap of X, and $\psi : R^+ \to R^+$ an increasing, continuous function satisfying

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property (2). Further, let a, b be two decreasing functions from $R^+/\{0\}$ into [0, 1[such that $a(t) + b(t) < 1/2$ for every $t > 0$.

Suppose that T satisfies the following condition :

(B) $$\psi(d(Tx, Ty)) \le a(d(x, y)) \{\psi(d(x, y)) + c[\psi(d(x, y)) \psi(d(y, Tx)]^{1/2}\} + b(d(x, y) \{\psi(d(x, Tx)) + \psi(d(y, Ty))$$

where $x, y \in X$ and $c \in [0, 1]$ such that $a(t).(1 + c) < 1$.

Then T has unique fixed point.

Now we establish the following theorem :

Theorem 3. Let (X, d) be a complete metric space. T is a selfmap of X, and $\psi : R^+ \to R^+$ an increasing continuous function satisfying property (2). Further let a, b be two decreasing functions from$R^+/\{0\}$ into [0, 1 [such that $a(t) + b(t) < 1$ for every $t > 0$. Suppose that T satisfies the following condition :

(A)′ $$\psi(d(Tx, Ty) \le a(d(x, y)) \max \{\psi(d(x, y)), [\psi(d(x, y). \psi(d(y, Tx))]^{1/2}\} + b(d(x, y) \min \{\psi(d(x, Tx), (\psi(d(y, Ty))\}$$

where $x, y, \varepsilon X$. Then T has unique fixed point.

Proof: Let x_0 be a point of X. We define $x_{n+1} = Tx_n$, $\tau_n = d(x_n, x_{n+1})$ for all $n \in N \cup \{0\}$. We first prove that T has a fixed point. We may assume $\tau_n > 0$ for each n.

From (A)′, we obtain

$$\psi(d(Tx_n, Tx_{n+1}))$$

$$\le a(d(x_n, x_{n+1})).\ \text{max.}\ \{\psi(d(x_n, x_{n+1})), [\psi(d(x_n, x_{n+1})).\ \psi(d(x_{n+1}, Tx_n))]^{1/2}\} + b(d(x_n, x_{n+1})) \min \{\psi(d(x_n, x_{n+1})), \psi(d(x_{n+1}, Tx_{n+1}))\}$$

$$\Rightarrow \quad \psi(d(x_{n+1}, x_{n+2})) \le a(\tau_n).\ \max \{\psi(\tau_n), [\psi(\tau n).\ \psi(d(x_{n+1}, x_{n+1})]^{1/2}\} + b(\tau_n) \min (\psi(\tau_n), \psi(\tau_{n+1})\}.$$

If $\psi(\tau_{n+1}) \le \psi(\tau_n)$,. then we obtain

$$\psi(\tau_{n+1}) \le a(\tau_n).\ \psi(\tau_n) + b(\tau_n).\ \psi(\tau_{n+1}).$$

(3.1) i.e. $$\psi(\tau_{n+1}) \le \frac{a(\tau_n).}{1 - b(\tau_n)} \psi(\tau_n) < \psi(\tau_n)$$

On the other hand, if $\psi(\tau_n) \le \psi(\tau_{n+1})$, then we have

(3.2) $$\psi(\tau_{n+1}) \le [a(\tau_n) + b(\tau_n)]\ \psi(\tau_n) < \psi(\tau_n)$$

Thus $\{\tau_n\}$ is a decreasing sequence since ψ is an increasing.

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In (3.1), we put $\lim_{n \to \infty} \tau_n = \tau$ and suppose that $\tau > 0$, then $\tau_n \geq \tau$ implies that

$$\psi(\tau_{n+1}) \leq \frac{a(\tau)}{1 - b(\tau)} \psi(\tau).$$

By letting $n \to \infty$, since ψ is continuous, we have

$$\psi(\tau) \leq \frac{a(\tau)}{1 - b(\tau)} \psi(\tau)$$

a contradiction. So $\tau = 0$.

Similarly if we consider (3.2) then we again have $\tau = 0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence, suppose not, then there exist $\epsilon > 0$ and two sequence $\{p^{(n)}\},\{q^{(n)}\}$ such that for every $n \in N \cup \{0\}$, we find that

$$p^{(n)} > q^{(n)} \geq n, d(x_p^{(n)}, x_q^{(n)}) \geq \epsilon \quad \text{and} \quad d(x_{p-1}^{(n)}, x_q^{(n)}) < \epsilon.$$

For each $n \geq 0$, we put $S_n = d(x_p^{(n)}, x_q^{(n)})$. Then, we have

$$\epsilon \leq S_n \leq d(x_{p-1}^{(n)}, x_p^{(n)}) + d(x_{p-1}^{(n)}, x_{q-1}^{(n)}) < \tau_{p-1}^{(n)} + \epsilon .$$

Since $\{\tau_n\}$ converges to 0, so that S_n converges to ϵ..

Furthermore, the triangle inequality implies, for each $n \geq 0$,

$$-\tau_p^{(n)} - \tau_q^{(n)} + S_n \leq d(x_p^{(n)} + 1, x_p^{(n)} + 1) \leq \tau_p^{(n)} + \tau_q^{(n)} + S_n,$$

and therefore also the sequence $d(x_{p+1}^{(n)}, x_q^{(n)} + 1)$ converges to ϵ..

Now from (A) we also deduce

$$\psi(d(x_{p+1}^{(n)}, x_{q+1}^{(n)}) \leq a(d(x_p^{(n)}, x_q^{(n)}).$$

$$\max \{\psi(d(x_p^{(n)}, x_q^{(n)}), [\psi(d(x_p^{(n)}, x_q^{(n)}) \cdot \psi(d(x_q^{(n)}, x_{p+1}^{(n)}))]^{1/2}\}$$

$$+ b(d(x_p^{(n)}, x_q^{(n)})) \min \{\psi (d(x_p^{(n)}, x_{p+1}^{(n)}), \psi(d(x_q^{(n)}, x_{q+1}^{(n)})$$

i.e. $\psi(d(x_{p+1}^{(n)}, x_{q+1}^{(n)}) \leq a(s_n) \max \{\psi(s_n), [\psi(s_n),$

$$\psi(s_n). \psi(d(x_q^{(n)}, x_{p+1}^{(n)}))]^{1/2}\} + b(s_n) \min \{\psi(\tau_p^{(n)}), \psi(\tau_q^{(n)})\}.$$

This implies that

$$\psi(d(x_{p+1}^{(n)}, x_{q+1}^{(n)}) \leq a(s_n) \max \{\psi(s_n), [\psi(s_n).$$

$$\psi((\tau_p^{(n)} + s_n)]^{1/2}\} + b\ (s_n) \min \{\psi(\tau_p^{(n)}), \psi(\tau_q^{(n)})\}.$$

Letting $n \to \infty$, we are left with

$$\psi(\epsilon) \leq a(\epsilon) \max \{\psi(\epsilon), [\psi(\epsilon).\psi(\epsilon)]^{1/2}\} + b(\epsilon) \min \{\psi(0), \psi(0)\}$$

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Therefore, $\psi(\in) \leq a(\in) \cdot \psi(\in) < \psi(\in)$ since $a(t) < 1$ which is absurd. Therefore $\{x_n\}$ is a Cauchy sequence.

By completness of X, $\{x_n\}$ converges to some point z. Now we show that z is a fixed point of T. Since $\tau_n > 0$ there is a subsequence $\{x_n(n)$ of $\{x_n\}$ such that $X_h(n) = z$ for each $n > 0$ and we put $\rho_n = d(z, x_n)$.

Since $a, b < 1$, we obtain from (A')

$$\psi(d(x_{h+1}(n), Tz)) = \psi \cdot (d(T x_h(n), Tz))$$

$$\leq a(d(Tx_h(n), z)) \max \{\psi(d(x_h(n),z)), [\psi(d(x_h(n), z)). \psi(d(z, x_{h+1}(n)))]^{1/2}\}$$

$$+ b((d(x_h(n), z) \min \{\psi(d(x_h(n), x_{h+1}(n))), \psi \{d(z, Tz))\}.$$

$$\Rightarrow \quad \psi(d(x_{h+1}(n), Tz))$$

$$\leq a(\rho_h(n)). \max \{\psi(\rho_h(n)), [\psi(\rho_h(n)). \psi(d(x_{h+1}(n), z))]^{1/2}\}$$

$$+ b(\rho_h(n)) \min \{\psi \tau_h(n), \psi(d(z, Tz))\}$$

Since $\{\rho_n\}$ converges to 0, letting $n \to \infty$, we have

$$\lim_{n \to \infty} \psi(d(x_{h+1}(n), Tz)) < 0. \tag{3.3}$$

On the other hand the triangle inequality implies that

$$d(z, Tz) \leq \rho h(n) + \tau_h(n) + d(x_{h+1}(n)), Tz),$$

which in turn implies that

$$\psi(d(z, Tz)) \leq \lim_{n \to \infty} \psi(d(x_{h+1}(n), Tz)). \tag{3.4}$$

Therefore from (3.3) and (3.4), we have

$\psi(d(z, Tz)) = 0$ and therefore $(d(z, Tz)) = 0$.

Then z is a fixed point of T.

If T has two distinct fixed points x, y in X, then

$$\psi(d(x, y) = \psi(d(Tx, Ty))$$

$$\leq a(d(x, y)) \max \{\psi(d(x, y)), [\psi(d(x, y). \psi(d(y, Tx))]^{1/2}\}$$

$$+ b(d(x, y)) \min \{\psi(d(x, Tx)), \psi(d(y, Ty))\}$$

$$\Rightarrow \quad \psi(d(x, y) \leq a(d(x, y)) \max \{\psi(d(x, y)), \psi(d(x, y))\},$$

since $Tx = x$ and $Ty = y \Rightarrow \psi(d(x, y)) < \psi(d(x, y))$ a contradiction. This completes the proof.

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FUNDAMENATAL SOLUTION IN GENERALIZED FUNCTION SPACE IN NEGATIVE RESISTANCE OSCILLATORY CIRCUIT OF DYNATRON OSCILLATOR

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ABSTRACT

Using the method of variation of parameters and a special case of S.M. Joshi Distributional transform, fundamental solutions in generalized function space are utilized to analyze the electronic circuit of Dynatron Oscillator. The appropriate conditions for sustained and decaying oscillations are obtained by the new method.

If the decrease in the applied voltage induced increase in the current, in electronic circuitary arrangement, the resistance offered by this arrangement is called negative resistance. For crimple in a Vacuum tetrode the secondary commission displays the negative resistance. Herold [2] developed the idea (see Fig.1) that the sustained oscillations of high frequency stability can be achieved by shunting a tuned circuit with the negative resistance (here tetrode). The oscillator operators over the negative resistance region EF of tetrode plate characteristics. In this region the screen grid voltage becomes much higher than plate voltage for a larger number, than that of primary electrons, of secondary electrons may be produced by the plate and collected by the screen grid. Plate voltages e_{b_1} and e_{b_2} are kept suitably below the screen voltage to offer the negative resistance, due to secondary electron emmission from the plate. The positive resistance of the circuit is neutralized by the negative resistance offered by the tube.

The commercial application of negative resistance is made in dynatron, killitron and transitron oscillators. the dynatron (see Fig. 2), for example, oscillator operates in the negative resistance region of tetrode.

The present paper is concerned with analyzing mathematically the above circuit using the powerful method of fundamental solutions.

Using Kirchoff's law in the equivalent circuit of dynatron oscillator, differential equation for current I_t, after some simplification, takes the form

$$[D^2 + \mu D + \nu^2]\, I_t = 0, \qquad \text{where } D \text{ stands for the operator } d/dt, \tag{1}$$

$$\mu = \frac{\gamma_p RC - L}{\gamma_p \,.\, RC} \quad \text{and} \quad \nu^2 = \frac{\gamma_p - R}{\gamma_p \,.\, LC}\,. \tag{2}$$

Although one can use the method of variation of parameters [4] to solve (1) to study the nature of oscillatory current flowing in a dynatron oscillator as has been done in the following section yet that does not at least mathematically, give the whole picture, we therefore obtain the solution with the help of distributional transform

$$(S^a_{b_0} f)(x) = \langle f(\xi), e^{r\xi x} \rangle, \tag{3}$$

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which is a special case of distributional S.M.Joshi tranform [3]

(4) $$(s_b^a f)(x) = \langle f(\xi), e^{i\xi x} \rangle > 0.$$

Notation and terminology. Here

(5) $$\langle f, \phi \rangle$$

is taken to mean the application of the functional f to the test function ϕ, and $\mathcal{F}$ is the space of basic functions i.e. the space of infinitely differentiable functions which together with all their derivatives approach zero more rapidly than any power of $1/|x|$ as $x \to \infty$.
The generalized function (or the distributions as they are called) Space on T i.e. the set of all linear continuous functionals on $\mathcal{F}$ is denoted by T' as dual space of T.

The linearity of is $\mathcal{F}$ signified by the relation

(6) $$\langle f, \alpha_1\phi_1 + \alpha_2\phi_2 \rangle = \alpha_1 \langle f, \phi_1 \rangle + \alpha_2 \langle f, \phi_2 \rangle$$

where as the continuity of the generalized function f implies that, if the sequence $\phi_1, \phi_2, \ldots, \phi_\nu, \ldots$ converges to zero in $\mathcal{F}$ then the sequence $\langle f, \phi_1 \rangle, \langle f, \phi_2 \rangle, \ldots$ also converges to zero in T'. For regular functionals

(7) $$\langle f, \phi \rangle = \int f(x)\, \phi(x)\, dx.$$

Functionals that can not be represented as (7), are called *singular*.

For example the delta functional represented by

(8) $$\langle \delta(x), \phi(x) \rangle = \phi(0)$$

is singular and so is the so called "translated" [1] delta functional is defined by

(9) $$\langle \delta(x - x_0), \phi(x) \rangle = \phi(x_0).$$

Sustained oscillations

To obtain solution of (1) by the method of fundmental solution let it has the driving impulse $-g(t, t')$ originating from a source point at the instant t'. We now concentrate on the solution of the differential equation

(10) $$[D^2 + \mu D + \gamma^2]\, I_t = -g(t, t').$$

The wellknown variation of parameter method then gives the solution as

(11) $$I_t = [A(t) \cos \rho t + B(t) \sin \rho t] \exp(\sigma t), \text{ if } 4\nu^2 > \mu^2$$

and

(12) $$I_t = [A(t) \cosh \rho t + B(t) \sinh \rho t] \exp(\sigma t), \text{ if } 4\nu^2 < \mu^2.$$

where

(13) $$\rho^2 = \frac{1}{2}(\mu^2 - 4\nu^2) \text{ and } \sigma = -\tfrac{1}{2}\mu$$

The variation of parameter method then requires, to obtain the parameters, that

(14) $$[\rho\{\frac{dB(t)}{dt} - (\tfrac{1}{2}\mu^2 - \nu^2) A(t)\} \cos \rho t$$

$$- \{\rho \frac{dA(t)}{dt} + (\tfrac{1}{2}\mu^2 - \nu^2) B(t)\} \sin \rho t] \exp(\sigma t) = -g(t, t')$$

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Solution could then be obtained from (11) and (14). But in practice it is difficult to be obtained unless we take

(15) $\mu^2 = 2\nu^2$ or in other words

(16) $$\left[\frac{\gamma_p RC - L}{\gamma_p LC}\right]^2 = 2\left[\frac{\gamma_p - R}{\gamma_p LC}\right].$$

This is exactly the condition of oscillations in Dynatron oscillators.

This gives $\gamma_p = \frac{1}{2C}\left(1 - \frac{R^2}{L}\right)^{-1}$, or $= \frac{1}{2C}\left(\frac{R^2}{L} - 1\right)^{-1}$.

Under the assumption of negative plate resistance, which as explained earlier in the fundamental requirement in the Dynatron oscillator, only the later value is edmissible. Hence

(17) $$w_0^2 R^2 C \leq 1,$$

where as usual w_0 is the resonant frequency.

The dumping factor μDI_t helps to decay the oscillations.

The sustained oscillations, then, can only be obtained if

(18) $\mu = 0$ and $\gamma_p = L/CR$.

The frequency of oscillation is, then, given by

(19) $$f = \frac{LC}{2\pi} - \frac{1}{2}\left(1 - \frac{R}{\gamma_p}\right)^{1/2}.$$

Equation (1) under these conditions is that of simple Harmonic oscillator. However practically such an adjustment is difficult to obtain.

Fundamental Solution in τ'.

A fundamental solution in τ' is the solution of the equation

(20) $$[D^2 + \mu D + \nu^2]\, F(t, t') = -\delta(t, t'),$$

where $\delta(t, t')$ is the "translated" generalized delta function defined by [1], of unit impulse.

We take the timings t and t' when the oscillator operates at points E and F of Fig.2 respectively.

Once $F(t, t')$ is known the solution for oscillatory current equation (10) can be obtained [1] as

(21) $$I_t = F(t) * g(t),$$

where * denotes convolution operation in T'.

But taking Distributional transform (4) of both sides of (20) one obtains

$$F(t, t') = \frac{1}{\xi^2 + \mu\xi - \nu^2}, \qquad \text{where } F(t) = (s_{b_0}^{a_0} F).$$

Taking inverse [3] we easily obtain

(22) $$F(t, t') = \frac{1}{2\pi} < (\rho^2 + i\mu\xi - \nu^2)^{-1}, exp\ t\ \xi(t - t') >$$

Hence

(23) $$F(t, t') = \{2(4\nu^2 - \mu^2)^{-1/2} \sin [(\nu^2 - \mu^2/4)^{1/2} (t - t')]\ expt\ (t - t')^{1/2}\}$$

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$$\{H(t-t')\},$$

where $H(t-t')$ is the Heavisides unit step Function.

Discussion

To obtain sustained oscillations it is necessary that γ_p be given as closely as given by (18) so that harmonically oscillating system of oscillating frequency 'f' (given by (19) may be function.

This is verified by the forms of solution (23) which shows that the oscillation dies out exponentially resulting in failure of dynatron oscillator.

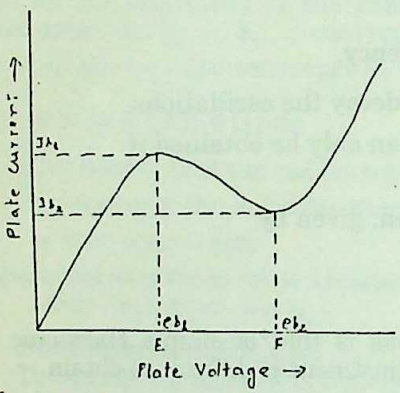

FIG 1 TETRODE CHARACTERISTIC

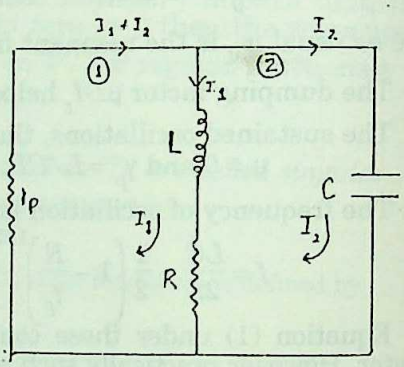

FIG 2(b) - EQUIVALENT CIRCUIT OF DYNATRON.

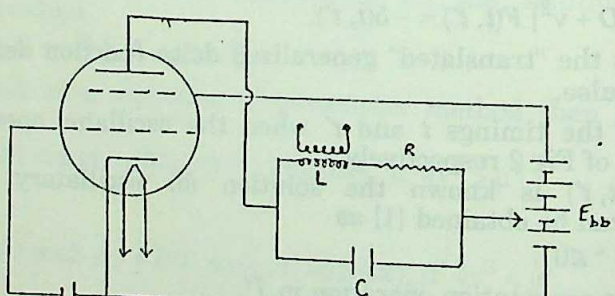

FIG 2(a) DYNATRON OSCILLATOR

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SCATTERING IN GENERALIZED FUNCTION SPACE

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1. INTRODUCTION

The most direct information about the nature of forces between the particles is obtained from the study of collisions. The experimental data can be interpreted in terms of the model of the miscroscopic details of a single collision and the conclusions concerning the interaction between the scattered particle and the scatter can be drawn. In recent years this study has assumed quite importance because of its wide applications.

Attempts have been made [1, 4, 5] to analyse this type of scattering by several authors but all of them have givn aproximate methods and adoption of these analytical theories makes it difficult to choose the appropriate value of parametres thereby restricting the proper choice of energy values of the system and the symmetry of the potential function.

The present paper attempts to solve the corresponding differential equation in a general approximation free way in the so called generalized function space (or distribution space). The technique adopted substantially expands the range of problems that can be tackled.

In what follows it is assumed that the interaction between the scattered particle and the scatterer can be represented by the potential energy function $V(\boldsymbol{r})$, where $\boldsymbol{r}$ is the vector joining the scattered particle and the centre of force. At present we restrict ourselves to elastic scattering only in which the kinetic energy of the system is not changed due to the collision. If the mass of the scatterer is large as compared to that of scattered particle e.g. in the collision between an electron and an atom the scatterer can be assumed to remain at rest during the entire collision process.

2. Notation and Terminology

A generalized function is a generalization of the classical notion of a function. It reflects the fact that in reality one can not measure the value of a physical quantity at a point, but can only measure the mean values within sufficiently small neighbourhoods of the point and

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then proclaim the limit of the sequence of those mean values as the value of the physical quantity at the given point. Thus it enables one to express in regorous mathematical form such ideal concepts as density of a point charge or dipole and the Paul Dirac's delta function.

We shall be interested in finding the solution of the differential equation of the scattering process discussed in the last para of Section 1, viz.

$$(2.1) \quad (\nabla^2 + k^2) = \frac{2m}{h^2} V(\boldsymbol{r}),$$

where $k^2 = \frac{2m}{h^2} E$

in the space $\tau[2]$ of basic functions i.e. the space of infinitely differentiable functions which, together with their derivatives, aproach zero more rapidly than any power of $1/|x|$ as $|x| \to \infty$. The generalized function space of τ i.e. the set of continuous linear functionals on τ shall be denoted by τ' as dual space of τ. The space τ' is known as the space of generalized functions of slow growth and is very important in applications of Mathematical Physics. The terminology used here is that of [2].

3. Formulation of the problem

In general, the potential energy $V(\boldsymbol{r})$ decreases in magnitude as the distance $r = |\hat{\boldsymbol{r}}|$ from the scattering centre becomes large, it is convenient to choose the arbitrary constant in the definition of $V(\boldsymbol{r})$ such that $V = 0$ at $r = \infty$.

If $V(r)$ decreases to zero sufficiently rapidly as $r \to \infty$, the particle can be considered essentially free when r is large. The asymptotic wave function is therefore, in general, a linear superposition of the free particle wave function $exp\ (i\boldsymbol{k}.\boldsymbol{r})$.

A solution of (2.1) is sought which satisfies the boundary condition

$$(3.1) \quad \psi \sim \exp(i\,\hat{\boldsymbol{k}}.\,\hat{\boldsymbol{r}}) + f(k, k')\, eikr/r$$

,where r is large enough for the particle to be beyond the range of force. The scattering amplitude is $f(k, k')$, where k and k' are propogation vectors.

It is convenient to separate the incidental wae function $exp\ (i\ \boldsymbol{k}.\ \boldsymbol{r})$ from the wave function and to write

$$(3.2) \quad \psi = \exp(i\,\hat{\boldsymbol{k}}\,.\,\hat{\boldsymbol{r}}) + \psi_s$$

where ψ_s the scatterer wave, is asymptotic to e^{ikr}/r. Since $\exp(i\,\hat{\boldsymbol{k}}\,.\,\hat{\boldsymbol{r}})$ is a solution of homogeneous wave equation

$$(3.3) \quad (\nabla^2 + k^2) \exp(i\,\hat{k}\,.\,\hat{r}) = 0,$$

the corresponding Schrodinger's equation takes the inhomogeneous form

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(3.4) $(\nabla^2 + k^2)\,\psi_s = \frac{2m}{h^2} V(r)\psi.$

A formal solution of (3.4) can now be obtained in generalized function space τ'.

4. The solution in the space τ' .

Let us take

(4.1) $\frac{2m}{h^2} - V(r)\psi = -\rho(r)$

Here the quantity $\rho(r)$ can be regarded as a source density function for scattered divergent spherical waves.

Before finding the actual solution it may be pointed out that if $\psi(s_1)$ and $\psi(s_2)$are solutions of (3.7), with $\rho(r)$ as in (4.1), belonging to density functions $\rho_1(r)$ and $\rho_2(r)$ satisfying

(4.2) $< \psi s_1, \phi > < f_1, e^{ikr}/r, \phi>$, and

(4.3) $< \psi_2 \phi > \sim < f_2\, e^{ikr}/r, \phi>$

respectively; then $\psi_s = \psi_{s_1} + \psi_{s_2}$ is a soluiton belonging to $f(r) = \rho_1(r) + \rho_2(r)$ such that

(4.4) $< \psi_S, \phi > \sim < f, e^{ikr}/r, \phi>,$

where $f = f_1 + f_2$ is a well defined linear continuous functional defined over the space of density functions $\rho_1(r)$ and $\rho_2(r)$.

Here

(4.5) $< f, \phi >$

is taken to mean as the application of the functional $f \in \tau'$ to the test function $\phi \in \tau$.

Also the linearity of f is signified by the relation

(4.6) $< f, \alpha_1\phi_1 + \alpha_2\phi_2> = < \alpha_1 ; f, \phi_1 > + < \alpha_2\ f, \phi_2>$

where as continuity of the generalized function f implies that if the sequence $\phi(, \phi_2, ..., \phi_\nu, ...$ converges to zero in τ then the sequence $<f, \phi_1>, <f, \phi_2>$, ..., $<f_1, \phi_\nu>$, ... also converges to zero.

For *regular* functionals

4.7 $< f, \phi> = \int f(x)\, \phi\, (x)\, dx.$

Functionals that can not be represented as (4.7) are called *singular.*

For example the delta functional represented by

(4.8) $< \delta(x), \phi(x) > = \phi(0)$

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is singular, and so is the socalled "translated" delta functional defined by

(4.9) $< \delta (x - x_0), \phi(x)> = \phi(x_0)$.

Now to obtain the desired solution let $E(r, r')$ be the solution of the equation

(4.10) $(\nabla^2 + k^2)\, E(r, r') = -\, \delta(r, r')$,

where $E(r, r')$ is asymptotic to e^{ikr}/r.

Here $\delta(r, r')$ is the source density of unit strength.

Using the distributional Fourier transform technique, we have

(4.11) $E(r, r') = \dfrac{1}{2\pi} < E(r, r'), e^{-ik(r-r')\xi} >$,

where $E(r, r')$ denotes the distributional Fourier transform of

$E(r, r')$ *i.e.* $E(r, r') = 1/k^2 - \xi^2$, using (4.10).

The solution for the scattering problem for the density function $\rho(r)$ can then be calculated from the equation

(4.12) $\psi_s = \rho(r)^* E(r, r')$.

Here * denotes the convolution defined by

(4.13) $< f * g, \phi > = < f(x) \times g(y), \phi (x + y)>$.

For example

$< \delta * f, \phi > = < \delta(x) \times f(y), \phi(x + y) >$

$= < f(y), < \delta (x), \phi(x + y)>> = < f(y), \phi(y)> = < f, \phi>$.

Thus for any generalized function f and the generalized delta function defined by (4.9), we have

(4.14) $\delta * f = f * \delta = f$

and that, in general,

(4.15) $f * g = g * f$

holds also for generalized functions f and g in which the convolution operation has a meaning.

Now, using (4.9), the arbitrary density $\rho(r)$ is given by

(4.16) $\rho(r) = < \delta(r, r'),\ \rho(r')>$,

where $\rho(r')$ is the density function of point source at r'.

The scattering problem is thus formulated in the form of an integral equation, by (4.1) and (3.2)

(4.17) $\psi(r) = \exp (i \hat{\boldsymbol{k}} \,.\, \hat{\boldsymbol{r}}) - \dfrac{2m}{h^2} < E(r, \boldsymbol{r}'), V(r')\ \psi (r')>$

5. Discussion

Equation (4.17) shows that the scattered wave at the point r is composed of spherical waves $E(r, r')$ originating at each point of space

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r. The amplitude of each contribution is proportional to the product $V(r')\,\psi(r')$ i.e. it is porportional jointly to the strength of the interaction and the amplitude of wave function at r'. All these spherical waves are compounded at the point r and generate the total scattered wave, which is then added to the incident wave to produce the total wave function at r.

Also, if the potential energy function is assumed to be confined to a limited region of space, the asymptotic form of $E(r, r')$ as indicated at (4.10) can be substituted in (4.17) to yield the scattering amplitude

$$(4.18)\quad f(k, k') = -\frac{2m}{h^2} < \exp(-i\boldsymbol{k} \cdot \boldsymbol{r}'), V(r')\,\psi(r') >$$

Equation (4.18) is a general formula by which the scattering cross section $|f(k, k')|^2$ can be computed.

Conclusion

The solution thus achieved are more general in nature than those found in the existing literature as they are neither subjected to any restriction on the energy values of the system nor on the symmetry the potential function. More over the results may find applications in the study of matter in bulk. Although, (4.17) does not provide an explicit solution as appears inside, yet the formulation of the problem according to (4.17) provides a single equation which expresses both the content of the differential equation (2.1) and the boundry condition (3.1).

Also the method applies equally suitably, if instead of the distributional Fourier transform one takes its wider generalization (introduced by the first author recently in 3), called S.M. Joshi generalized Fourier transform, given by

$$(5.1)\quad (S_b^a f)(x) = \frac{\Gamma(a)\,\Gamma(b-\alpha)}{\Gamma(b)\Gamma(a-\alpha)} < f(\xi), {}_1F_1(a;b;i\,\xi\,x) >,$$

where $a = \alpha + \lambda + 1$, $b = a + \mu$, and ${}_1F_1$ is the confluent hypergeometric function. For $\mu = 0 = \alpha$, (5.1) reduces to the distributional Fourier transform

$$(5.2)\quad (S_{b_0}^{a_0} f)(x) = < f, e^{i\xi x} > .$$

Last, but the most fruitful, advantage of the method is the fact that it applies to both regular (the classical one) and singular functional, as pointed out in (4.7) and the line following it, and it utilizes the most modern concept of generalized delta function, which existed in the literature previously only in mathematically non rigorous form-a form about which its originator Paul Dirac (See "The Principles of Quantum Mechanics, "Clarendon Press, Oxford (1958) by Dirac) himself had doubts that whether it was a function in the classical meaning.

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INTEGRATION OF *I*-FUNCTION OF TWO VARIABLES WITH RESPECT TO PARAMETERS

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ABSTRACT

The object of the present paper is to establish four integrals associated with *I*-function of two variables due to Goyal and Agrawal [3]. The integration is performed with respect to a parameter. Such integrals are usuful in the study of certain boundary value problems.

1. INTRODUCTION

The *I*-function of two variables represented by means of two contour integrals is defined by Goyal and Agarwal [3] and represented in as following manner :

$$I\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{1}{(2\pi w)^2} \int_{l_1}\int_{l_2} \Phi_1(\xi)\, \Phi_2(\eta)\, \Psi(\xi,\eta)\, z_1^{\xi} z_2^{\eta}\, d\xi\, d\eta$$

where

$$w = \sqrt{-1},$$

$$\Phi_1(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n_2} (-a_j + \alpha_j \xi)}{\sum_{i=1}^{r} \left[\prod_{j=m_2+1}^{q_i^{(1)}} \Gamma(1 - b_{ji} + \beta_{ji}\xi) \prod_{j=n_2+1}^{p_i^{(1)}} \Gamma(a_{ji} - \alpha_{ji}\xi) \right]} \quad \ldots (1.2)$$

$$\Phi_2(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(d_j - \delta_j \eta) \prod_{j=1}^{n_3} \Gamma(1 - c_j + \gamma_j \eta)}{\sum_{i=1}^{r} \left[\prod_{j=m_3+1}^{q_i^{(2)}} \Gamma(1 - d_{ji} + \delta_{ji}\eta) \prod_{j=n_3+1}^{p_i^{(2)}} \Gamma(c_{ji} - \gamma_{ji}\eta) \right]} \quad \ldots (1.3)$$

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$$\Psi(\xi,\eta) = \frac{\prod_{j=1}^{m_1}\Gamma(f_j - F_j(\xi+\eta))\prod_{j=1}^{n_1}\Gamma(1-e_j+E_j(\xi+\eta))}{\prod_{j=m_1+1}^{q}\Gamma 1-f_j+F_j(\xi+\eta\prod_{j=n_1+1}^{p}\Gamma e_j - E_j(\xi+\eta)} \quad \ldots (1.4)$$

z_1 and z_2 both are not equal to zero.

$$A = \sum_{j=1}^{n_1} E_j - \sum_{j=n_1+1}^{p} E_j + \sum_{j=1}^{m_1} F_j - \sum_{j=m_1+1}^{q} F_j + \sum_{j=1}^{m_2} \beta_j - \sum_{j=m_2+1}^{q_i^{(1)}} B_{ji}$$

$$+ \sum_{j=1}^{n_2} \alpha_j - \sum_{j=n_2+1}^{p_i^{(1)}} \alpha_{ji} > 0 \quad \ldots (1.5)$$

$$B = \sum_{j=1}^{n_1} E_j - \sum_{j=n_1+1}^{p} E_j + \sum_{j=1}^{m_1} F_j - \sum_{j=m_1+1}^{q} F_j + \sum_{j=1}^{m_3} \delta_j - \sum_{j=m_3+1}^{q_i^{(2)}} \delta_{ji}$$

$$+ \sum_{j=1}^{n_3} \gamma_j - \sum_{j=n_3+1}^{p_i^{(2)}} \gamma_{ji} > 0 \quad \ldots (1.6)$$

$\forall\ i = 1, 2, \ldots r$

$$|\arg z_1| < \frac{A\pi}{2}, \quad |\arg z_2| < \frac{B\pi}{2}.$$

For asymptotic expansion and all other conditions, see Goyal Anil and Agrawal R.D. [3]. These conditions will be assumed to hold good in the I-function of two variables occuring in this paper.

2. FORMULAE REQUIRED

In this paper, we require the following results

It is a hypergeometric function with unit argument given by Whittakar and Watson [2]

$$\frac{1}{2\pi w}\int_{-i\infty}^{+i\infty} \frac{\Gamma(a+x)\,\Gamma(b-x)\,\Gamma c-x)}{\Gamma(d-x)}\, e^{\pm i\pi x}\, dx$$

$$= \frac{\Gamma(a+b)\,\Gamma(a+c)\,\Gamma(d-a-b-c)}{\Gamma(d-b)\,\Gamma(d-c)}\, e^{\pm i\pi a}$$

$$\mathrm{Re}(d-a-b-c) > 0 \quad \ldots (2.1)$$

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1), & n \geq 1 \\ 1 \quad \text{for} \quad n = 0, & \alpha \neq 0 \end{cases} \quad \ldots (2.2)$$

3. INTEGRALS

The integrals involving the I-function of two variables to be evaluated are

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(I) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma b-x\,\Gamma c-x}{\Gamma d-x}\,e^{\pm i\pi x}\;I^{m_1,\,n_1+1:P}_{p+1,\,q+1:Q}\left[\begin{matrix}z_1\\ z_2\end{matrix}\right.\left|\begin{matrix}(1-a-x:h,k),(e_p,E_p):T\\ (f_q,F_q)\qquad\qquad:T^1\end{matrix}\right]dx$$

$$=\frac{e^{\pm i\pi a}}{\Gamma\,d-b\,\Gamma\,d-c}\,I^{m_1+1,\,n_1+2:P}_{p+2,\,q+1:Q}\left[\begin{matrix}z_1e^{\pm i\pi h}\\ z_2^{\pm i\pi k}\end{matrix}\right.\left|\begin{matrix}(1-a-b:h,k),(1-a-c:h,k)\,(e_p,E_p):T\\ (d-a-b-c:h,k),(f_q,F_q)\qquad\qquad T^1\end{matrix}\right]dx$$

... (3.1)

provided $h,k>0$

$$\mathrm{Re}\left[d-a-b-c+h\;\min_{1\le j\le m_2}R\left(\frac{b_i}{\beta_j}\right)+k\;\min_{1\le j\le m_3}R\left(\frac{d_j}{\delta_j}\right)\right]>0$$

... (3.1.1)

$$P=m_2,n_2;m_3,n_3,\quad Q=p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r \qquad \text{... (3.12)}$$

$$T=[(a_j,\alpha_j)_{1,n_2}],[(a_{ji},\alpha_{ji})_{n_{2+1}},p_i^{(1)}];[c_j,\gamma_j)_{1,n_3},[(c_{ij},\gamma_{ji})_{m_3+1},p_i^{(2)}]$$

... (3.13)

$$T^1=[(b_j,\beta_j)_{1,m_2},[(b_{ji},\beta_{ij})_{m_2+1,q_i^{(1)}}];[(d_j,\delta_j)_{1,m_3}],[(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}}]$$

... (3.14)

(II) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\Gamma(a+x)\,\Gamma(b-x)\,\Gamma\,c-x\,e^{\pm i\pi x}$$

$$I^{m_1,\,n_1:P}_{p+1,\,q:Q}\left[\begin{matrix}z_1\\ z_2\end{matrix}\right.\left|\begin{matrix}(e_p,E_p),(d-x:h,k):T\\ (f_q,F_q)\qquad\qquad:T^1\end{matrix}\right]dx$$

$$=e^{\pm i\pi a}\Gamma(a+b)\,\Gamma\,(a+c)\,I^{m_1+1,\,n_1:P}_{p+2,\,q+1:Q}\left[\begin{matrix}z_1\\ z_2\end{matrix}\right.\left|\begin{matrix}(e_p,E_p),(d-b:h,k),(d-c:h,k):T\\ (d-a-b-c:h,k),(f_q,F_q))\qquad:T^1\end{matrix}\right]$$

... (3.2)

provided $h,k>0$

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$$\operatorname{Re}\left[d-a-b-c-h \quad \min_{1\le j\le m_2} \operatorname{Re}\left(\frac{b_i}{\beta_j}\right)+k \quad \min_{1\le j\le m_3} \operatorname{Re}\left(\frac{d_j}{\delta_j}\right)\right]>0 \qquad \ldots (3.2.1)$$

(III) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \Gamma(a+x)\,\Gamma(b-x)\,\Gamma(c-x)\,e^{\pm i\pi x}$$

$$I^{m_1,\,n_1+1:P}_{p+2,\,q:Q}\left[z_1 \;\middle|\; \begin{matrix}(1-c+x:h,k),(e_p,E_p),(d-x;h,k):T\\ (f_q,F_q) \qquad\qquad :T^1\end{matrix}\right]dx$$

$$= e^{\mp i\pi a}\Gamma(a+b)\,I^{m_1+n_1+1:P}_{p+3,\,q+1:Q}\left[z_1 \;\middle|\; \begin{matrix}(1-a-c:h,k),(e_p,E_p),(d-b:h,k),(d-c:2h,2k):T\\ (d-a-b-c:2h,2k),(f_q,F_q)) \qquad\qquad :T^1\end{matrix}\right] \qquad \ldots (3.3)$$

provided $h, k > 0$

$$\operatorname{Re}\left[d-a-b-c-2h \quad \min_{1\le j\le m_2} R\left(\frac{b_j}{\beta_j}\right)-2k \quad \min_{1\le j\le m_3} R\left(\frac{d_j}{\delta_j}\right)\right]>0 \qquad \ldots (3.3.1)$$

(IV) $$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(a+x)\,\Gamma(b-x)}{\Gamma(c-x)}\,e^{\pm i\pi x}\; I^{m_1,\,n_1+1:P}_{p+1,\;\;q\;:Q}\left[z_1 \;\middle|\; \begin{matrix}(1-d-x:h,k),(e_p,E_p):T\\ (f_q,F_q) \qquad\qquad :T^1\end{matrix}\right]dx$$

$$= e^{\pm i\pi a}\frac{\Gamma(a+b)}{\Gamma(c-b)}\,I^{m_1+1,\,n_1+1:P}_{p+2,\,q+1:Q}\left[z_1 \;\middle|\; \begin{matrix}(1-a-d):h,k),(e_p,E_p),(c-d:h,h,k):T\\ (c-a-b-d:h,k),(f_q,F_q) \qquad\qquad T^1\end{matrix}\right]dx \qquad \ldots (3.4)$$

provided $h, k > 0$

$$\operatorname{Re}\left[c-a-b-d-h \quad \min_{1\le j\le n_2} R\left(\frac{a_j-1}{\alpha_j}\right)-k \quad \max_{1\le j\le n_3} R\left(\frac{c_j-1}{\gamma_j}\right)\right]>0 \qquad \ldots (3.4.1)$$

Here P, Q, T, T^1 are defined as per equations (3.1.2), (3.1.3), (3.1.4) respectively.

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Proof. To prove the integral (3.1), we replace *I*-function of two variables by (1.1), change the order of integration which is justified under the conditions stated in (1.5), (1.6). We get

$$\frac{1}{(2\pi\omega)^2}\int_{i_1}\int_{i_2}\Phi_1(\xi)\,\Phi_2(\eta)\,\psi(\xi,n)\,z_1^{\xi}\,z_2^{\eta}$$

$$=\left\{\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma b-x\,\Gamma a-x+h\xi+k\eta\,\Gamma c-x}{\Gamma d-x}\,e^{\pm i\pi x}\,dx\right\}d\xi\,d\eta \quad \ldots (3.5)$$

Now we evaluate the inner integral using a known integral formula given by Whittakar and Watson [2], noting that it is a hypergeometric function with unit argument.

On expressing the resulting expression with the help of (1.1) we obtain (3.1).

Proceeding similarly we easily evaluate the integrals (3.2), (3.3), (3.4) respectively.

PARTICULAR CASES

(i) If we take $r = 1$, all greek letters equal to unity in (1.1), the *I*-function of two variables would reduce to relatively more familiar *G*-function of two variables. Thus our result (3.1), (3.2), (3.3), (3.4) will yield similar integrals involving the *G*-function of two variables.

(ii) If we set $r = 1, m_1 = 0$ in (3.1), (3.2), (3.3), (3.4), we get integrals for the *H*-function of two variables.

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COMMON FIXED POINT OF TWO PAIRS OF COMMUTING MAPPING IN SAKES SPACE

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ABSTRACT

Meir and Keeler [1] obtained a remarkable generalisation of Banach contraction principle. In this paper, a common fixed point theorem for the pairs of commuting mapping in a Saks space, satisfying Meir and Keeler type condition is obtained.

Theorem Let P and S be commuting mappings and Q and T be commuting mappings of a Saks space $(Xs.d) = (X.N_1.N_2)$ into itself satisfying the following conditions :

Given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, ($\delta(\epsilon)$ being a non decreasing function of ϵ) such that for all $x, y \in X, \epsilon \leq K \max \{N_2(Sx - Ty), N_2(Px - Sx), N_2(Qy - Ty), N_2(Px - Ty), \quad N_2(Sx - Qy)\} < \epsilon + \delta$

$$\Rightarrow N_2(Px - Qy) < \epsilon \qquad \ldots (1)$$

$$Px = Qy \text{ whenever } Px = Sx, Qy = Ty \qquad \ldots (2)$$

where $0 \leq 2K < 1$.

Further if the range of T contains the range of P and the range of S contains the range of Q and if one of P, Q, S and T is continuous then P, Q, S and T have a common fixed point z. Further z is the unique common fixed point of P and S of Q and T.

Proof : For the proof of this theorem we recall the lemma of Orlicz [2] for a Saks space.

Lemma Let $(Xs, d) = (X, N_1, N_2)$ be a Saks space. Then the following statements are equivalent.

(*i*) N_1 is equivalent to N_2 on X

(*ii*) (X, N_1) is a Banach space and $N_1 \leq N_2$ on X

(*iii*) (X, N_2) is a Frechet space and $N_2 \leq N_1$ on X.

Now we shall prove the theorem. First with the help of (1), we note that for all x, y in X such that

$$Px \neq Sx, Qy \neq Ty, N_2(Px - Qy) < K \max \{N_2(Sx - Ty), N_2(Px - Sx) N_2(Qy - Ty), N_2(Px - Ty), N_2(Sx - Qy)\}.$$

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Next the non decreasing character of $\delta(\epsilon)$ implies that given $\epsilon > 0$, there exists $\epsilon > 0$ such that

$$\epsilon' < \epsilon < \epsilon' + \delta(\epsilon') \text{ or equivalently}$$
$$K \max \{N_2(Sx - Ty), N_2(Px - Sx), N_2(Qy - Ty), N_2(Px - Ty),$$
$$N_2(Sx - Qy)\} = \epsilon \qquad \Rightarrow N_2(Px - Qy) < \epsilon', \epsilon' < \epsilon \qquad \ldots (4)$$

Let x_0 be an arbitrary point in X Choose a point x_1 and then a point x_2 in X such that $Px_0 = Tx_1$ and $Qx_1 = Sx_2$.

This can be done as $PX \subset TX$. In general having chosen the point x_{2n} choose a point x_{2n+1} and then a point x_{2n+2} such that

$$Px_{2n} = Tx_{2n+1} \text{ and } Qx_{2n+1} = Sx_{2n+2}.$$

Then we have,

$$N_2(Px_{2n} - Qx_{2n+1}) < K \max \{N_2(Sx_{2n} - Tx_{2n+1}), N_2(Px_{2n} - Sx_{2n}),$$
$$N_2(Qx_{2n+1} - Tx_{2n+1}), N_2(Px_{2n} - Tx_{2n+1}, N_2(Sx_{2n} - Qx_{2n+1})\},.$$

$$N_2(Px_{2n} - Qx_{2n+1}) < K \max \{N_2(Qx_{2n-1} - Px_{2n}), N_2(Px_{2n} - Qx_{2n-1}),$$
$$N_2(Qx_{2n+1} - Px_{2n}\}, N_2(Px_{2n} - Px_{2n}), N_2(Sx_{2n} - Qx_{2n+1})\}$$

$$N_2(Px_{2n} - Qx_{2n+1}) < K \max \{N_2(Qx_{2n-1} - Px_{2n}), N_2(Qx_{2n-1} - Qx_{2n+1})\}$$
$$= K \max \{N_2(Qx_{2n-1} - Px_{2n}), N_2(Qx_{2n-1} - Px_{2n})$$
$$+ N_2(Px_{2n} - Qx_{2n+1})\}.$$

Or $N_2(Px_{2n} - Qx_{2n+1}) < K[N_2(Qx_{2n-1} - Px_{2n}) + N_2(Px_{2n} - Qx_{2n+1}\}$,

i.e. $N_2(Px_{2n} - Qx_{2n+1}) < K/(1-K)N_2(Qx_{2n-1} - Px_{2n})$.

Similarly we have

$$N_2(Qx_{2n-1} - Px_{2n}) = N_2(Px_{2n} - Qx_{2n-1}) < K/(1-K)N_2(Qx_{2n-1} - Px_{2n-2}).$$

From the last two inequalities we conclude that both $N_2(Px_{2n} - Qx_{2n+1})$ and $N_2(Qx_{2n+1} - Px_{2n+2}) = 0$ are monotonic decreasing sequences of positive real numbers.

Now $\lim_{n \to \infty} N_2(Px_{2n} - Qx_{2n+1}) = 0$

and $\lim_{n \to \infty} N_2(Qx_{2n+1} - Px_{2n+2}) = 0.$

For, if not, suppose for instance

$\lim_{n \to \infty} N_2(Px_{2n} - Qx_{2n+1}) = r$ where $r > 0$.

Then given $\delta > 0$ there exists a positive integer N such that for each integer $m > N$, we heave

$$r \le N_2(Px_{2m} - Qx_{2m+1}) < r + \delta \qquad \ldots (5)$$

Or $r \le K \max \{N_2(Sx_{2m+2} - Tx_{2m+1}), N_2(Px_{2m+2} - Sx_{2m+2}),$

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$$N_2(Qx_{2m+1}) - Tx_{2m+1}),$$
$$N_2(Px_{2m+2} - Tx_{2m+1}), N_2(Sx_{2m} - Qx_{2m+1})\} < r + \delta \quad \ldots (6)$$

Selecting δ in (6) in accordance with (1), for each $m \geq N$ we obtain $N_2(Px_{2m} - Qx_{2m+1}) < r$ and so, $N_2(Px_{2m+2} - Qx_{2m+3}) < r$ which contradicts (5). Therefore we have

$$\lim_{n \to \infty} N_2(Px_{2n} - Qx_{2n+1}) = 0$$

and similarly, $\lim_{n \to \infty} N_2(Qx_{2n+1} - Px_{2n+2}) = 0.$

Now by lemma,

$\lim_{n \to \infty} N_1(Px_{2n} - Qx_{2n+1}) = 0 = \lim_{n \to \infty} N_1(Qx_{2n+1} - Px_{2n+2})$. From these limits and by condition (1) it follows easily that $\{Px_0, Qx_1, Px_2, Qx_3, \ldots, Px_{2n}, Qx_{2n+1}, \ldots\}$ is a Cauchy sequence in the Saks space X and so has a limit z in X.

Hence the sub-sequence $\{Px_{2n}\} = \{Tx_{2n+1}\}$ and $\{Qx_{2n+1}\} = \{Sx_{2n+2}\}$ converge to the point z.

Let us now suppose that the mapping S is continuous. Then since the mappings P and S commute, the sequences $\{Sx_{2n}\}$ and $\{PSx_{2n}\}$ converge to the point Sz. Now we shall show that $Sz = z$. For if $z \neq Sz$, then $N_2(PSx_{2n} - Qx_{2n+1} < K \max\{S^2x_{2n} - Tx_{2n+1}),$
$N_2(PSx_{2n} - S^2x_{2n}), N_2(Qx_{2n+1} - Tx_{2n+1}), N_2(PSx_{2n} - Tx_{2n+1}),$
$N_2(S^2x_{2n} - Qx_{2n+1})\}.$

Letting $n \to \infty$ on both sides we have

$N_2(Sz - z) \leq K N_2(Sz - z)$ which is a contradiction. Hence we have $Sz = z$. Similarly,

$$N_2(Pz - Qx_{2n+1}) < K \max \{N_2(Sz - Tx_{2n+1}), N_2(Pz - Sz),$$
$$N_2(Qx_{2n+1} - Tx_{2n+1}), N_2(Pz - Tx_{2n+1}), N_2(Sz - Qx_{2n+1})\}.$$

Letting $n \to \infty$ on both sides, we get $z = Pz$.

Thus we have $z = Pz = Sz$. It shows that there exists a point z' in X such that $Sz = z = Pz = Tz'$, since the range of P is contained in the range of T.

Now, $QTz' = Qz = TQz'$ ... (A) as Q and T commute. Further $z = Qz'$ for

$$N_2(z - Qz') = N_2(Pz - Qz') < K \max \{N_2(Sz - Tz'), N_2(Pz - Sz),$$
$$N_2(Qz' - Tz'), N_2(Pz - Tz'), N_2(Sz - Qz')\} = K N_2(z - Qz'),$$

which implies $z = Qz'$, By (A) and $z = Qz'$, $Qz = Tz$. Thus $z = Qz = Tz$ for otherwise

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$$N_2(z - Qz) = N_2(Pz - Qz) < K \max \{N_2(Sz - Tz), N_2(Pz - Sz),$$
$$N_2(Qz - Tz)\, N_2(Pz - Tz), N_2(Sz - Qz)\}$$
$$= K N_2(z - Qz) \text{ which implies } z = Qz.$$

Thus we have $z = Pz = Sz = Qz = Tz$. So far we have proved that z is a common fixed point of P, Q, S and T. The proof is similar if T is continuous instead of S.

Now suppose that the mapping is continuous. Since P and S commute the sequences $\{P^2x_{2n}\}$ and $\{SPx_{2n}\}$ converge to Pz. So,

$$N_2(P^2x_{2n} - Qx_{2n+1}) < K \max \{N_2(SPx_{2n} - Tx_{2n+1}), N_2(P^2x_{2n} - SPx_{2n}),$$
$$N_2\, Qx_{2n+1} - Tx_{2m+1}, N_2(P^2x_{2n} - T\,x_{2n+1}), N_2(SPx_{2n} - Qx_{2n+1})\}$$

on letting $n \to \infty$ on both sides will lead to $z = Pz$. This shows there exists a point z' in X such that $z = Pz = Tz'$ as we have $z \in PX$ and since $PX \subset TX$, Then we have,

$$N_2(P^2x_{2n} - Qz') < K \max \{N_2\,(Sx_{2n} - Tz'), N_2(P^2x_{2n} - SPx_{2n}), N_2(Qz' - Tz'),$$
$$N_2(P^2x_{2n} - Tz'), N_2(SPx_{2n} - Q(z')\}$$

on letting $n \to \infty$ on both sides, we obtain $z = Qz'$

i.e. $Tz = TQz' = QTz' = Qz$. Further

$$N_2(Px_{2n} - Qz) < K \max \{N_2\,(Sx_{2n} - Tz), N_2(Sx_{2n} - Px_{2n}), N_2(Qz - Tz),$$
$$N_2(Px_{2n} - Tz), N_2(Sx_{2n} - Qz)\}$$

Again on letting $n \to \infty$ we get, $z = Qz = Tz$. This shows that there exists a point z'' in X such that $z = Qz = Sz''$ with the same reasoning that $QX \subset SX$. Now,

$$N_2(PZ'' - z) = N_2(Pz'' - Qz) < K \max \{N_2(Sz' - Tz),$$
$$N_2(Pz'' - Sz''), N_2(Qz - Tz), N_2(Pz'' - Pz), N_2(Sz'' - Qz)\}.$$

From which again we get $z = Pz'' = Sz''$. Since P and S commute $PSz'' = Sz = SPz'' = Pz = z$. Thus we have proved that z is again a common fixed point of P, Q, S and T. If the mapping Q is continuous instead of P, then the proof that z is again a comon fixed point of P, Q, S and T is similar.

Now we shall show the uniqueness of the point z. Suppose that w is also a common fixed point of P and S. Then,

$$N_2(w - z) = N_2(Pw - Qz) < K \max \{Sw - Tz), N_2(Pw - Sw), N_2(Qz - Tz),$$
$$N_2(Pw - Tz)\} = K N_2(w - z)$$

which leads us to conclude $w = z$. Thus z is the unique common fixed point of P and S. Similarly we can prove that z is the unique common fixed point of Q and T.

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ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE LIPSCHITZ CLASS BY $F(a, q)$ MEANS

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ABSTRACT

In the present paper, we obtain the degree of approximation of $f \in \text{Lip } \alpha\ (0 < \alpha \leq 1)$ by $F(a, q)$ means of its Fourier series.

1. INTRODUCTION

Let $C_{2\pi}$ be the space of all 2π - periodic and continuous functions defined on $[-\pi, +\pi]$, which is Banach space under "Sup" norm. For each $f \in C_{2\pi}$, let the Fourier series be given by

$$(1.1) \qquad f(x) \sim \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x).$$

We write

$$(1.2) \qquad \Phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}.$$

The family $F(a, q)$ of summability methods was introduced by Meir [5]. The summability matrix $\{C_{pk}\}$ belongs to $F(a, q)$ if it satisfies the following conditions :

p is a discrete or continuous parameter : $q = q(p)$ is a positive increasing function which tends to infinity as $p \to \infty$; 'a' is a positive constant : for every fixed $\delta : 1/2 < \delta < 2/3$

$$(1.3) \qquad C_{pk} = \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \{1 + 0\left(\frac{|k-q|+1}{q}\right) + 0\left(-\frac{|k-q|^3}{q^2}\right)\}$$

as $p \to \infty$ uniformly in k for $|k-q| \leq q^{\delta}$; and

$$(1.4) \qquad \sum_{|k-q| > q^{\delta}} (k+1) C_{pk} = 0\{\exp(-q^{\mu})\},$$

where μ is some positive number independent of p.

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Let

$$t_p(f:x) = \sum_{k=0}^{\infty} C_{pk}\, s_k(f:x) \tag{1.5}$$

denote the $F(a, q)$ mean of the Fourier series (1.1) of f, where $s_k(f:x)$ is is the kth partial sum of (1.1).

The family $F(a, q)$ contains the summability methods of generalised Borel, Euler, Taylor, S_α and (e, c).

It is known (see [4]) that

$$\sum_{k=0}^{\infty} C_{pk} = 1 + 0\,(q^{-1/2}). \tag{1.6}$$

The summability methods of Euler, Taylor, S_α and Borel satisfy (1.6) in the stronger form

$$\sum_{k=0}^{\infty} C_{pk} = 1. \tag{1.7}$$

2. Degree of approximation by borel means and (E, q) - means were obtained by Chandra [1] and [2] respectively. Extending the results of Chandra to (e, c) mean we [6] have proved the following theorems :

THEOREM A. Let $f \in C_{2\pi} \cap \text{Lip } \alpha$, $0 < \alpha \leq 1$. Then

$$\| t_n^c - f \| = 0\,(n^{-\alpha/2}), \tag{2.1}$$

where $t_n^c\,(f; x)$ is n^{th} $(e;, c)$ means of the Fourier series of f at x.

Since $F(a, q)$ method includes (e, c) method, it is natural to ask as to what will be the result if we apply $F(a, q)$ mean to obtain the degree of approximation for $f \in \text{Lip } \alpha\ (0 < \alpha \leq 1)$?

We shall prove the following theorem :

THEOREM : Let $[q]$ denote the integral part of $q = q(p)$ and $m = [q] + 1$. If $f \in \text{Lip } \alpha$, $0 < \alpha \leq 1$, then

$$\| t_p(f;x) - f(x) \| = \sup_{-\pi \leq x \leq \pi} | t_p(f:x) - f(x) | = 0(m^{-\alpha/2}). \tag{2.2}$$

3. For the proof of our theorem we shall need the following lemmas :

LEMMA 1. If $q = q(p)$, is an integer valued function of p, then, for $\frac{1}{2} < 6 < \frac{2}{3}$ we have

$$\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-q|} \leq q^{\delta} \sqrt{\frac{a}{\pi q}}\, \exp(-aq^{-1}(k-q)^2) \sin(k + 1/2)t\, dt$$

$$= \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \exp\left(-\frac{qt^2}{4a}\right) \sin\left(q + \frac{1}{2}\right)t\, dt + O(q \exp(-aq^{2\delta - 1})).$$

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PROOF OF LEMMA 1. From the proof of Lemma 3.2 of Ikeno [3] we have

$$\sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t$$

$$= \exp\left(-\frac{qt^2}{4a}\right) \sin\left(q+\frac{1}{2}\right)t\, dt + O(q \exp(-aq^{2\delta-1})\ |t|).$$

Hence the Lemma 1 follows in view of boundedness of $\Phi_x(t)$ and since $\sin(t/2) > t/\pi$ $(0 < t < \pi)$.

LEMMA 2. If m is as defined in the theorem, we have

$$\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$

$$= \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-m| \le m^{6}} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \sin(k+1/2)t\, dt$$

$$+ O(m^{-\alpha})$$

PROOF OF LEMMA 2. We estimate the difference :

$$\int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$

$$- \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{|k-m| \le m^{\delta}} \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \sin(k+1/2)t\, dt$$

$$= \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{m \le k \le m+m^{\delta}} \left[\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \right.$$

$$\left. - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m^2)) \right] \sin(k+1/2)t\, dt$$

$$+ \int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \sum_{m-m^{\delta} \le k < m} \left[\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \right.$$

$$\left. - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m^2)) \right] \sin(k+1/2)t\, dt$$

$$- \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{q+q^{\delta} < k \le m+m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin(k+1/2)t\, dt$$

$$+ \int_0^{\pi} \frac{\Phi_x(t)}{\sin t/2} \sum_{q-q^{\delta} \le k \le m-m^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$

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$$\sin (k + 1/2)t \, dt$$

(3.1) $$= D_1 + D_2 + D_3 + D_4.$$

From Ikeno [3], p. 261, 262, we have

$$\sum_{q+q^\delta < k \le m+m^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2).\sin(k+1/2)t \, dt$$

$$= O(\sqrt{q}\, t \exp(-aq^{2\delta-1}))$$

and also

$$\sum_{q-q^\delta \le k \le m-m^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)\sin(k+1/2)t \, dt$$

$$= O(\sqrt{q}\, t \exp(-aq^{2\delta-1}))$$

Thus,

$$|D_3| \le \int_0^\pi \frac{\|\Phi_x(t)\|}{t} \sum_{q+q^\delta < k \le m+m^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \, |\sin(k+1/2)t \,| \, dt$$

(3.2) $$= O(\sqrt{q} \exp(-aq^{2\delta-1})).$$

Similarly,

(3.3) $$|D_4| = 0(\sqrt{q} \exp(-aq^{2\delta-1})).$$

In case where $q < m \le k \le m + m^\delta$, we have

$$0 \le \frac{(k-m)}{\sqrt{m}} < \frac{(k-q)}{\sqrt{q}} < \frac{(k-[q])}{\sqrt{[q]}}.$$

Therefore, from Ikeno [3], p. 259

(3.4) $$\left| \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) - \sqrt{\frac{a}{\pi m}} \exp(-am^{-1}(k-m)^2) \right|$$

$$= O\{\frac{1}{\sqrt{m}} \exp(-am^{-1}(k-m)^2)(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m})\}.$$

Using (3.4), we obtain

$$|D_1| = O\ [\int_0^\pi \frac{|\Phi_x(t)|}{t} \sum_{m \le k \le m+m^\delta} \{\frac{1}{\sqrt{m}} \exp(-am^{-1}(k-m)^2)$$

$$(\frac{(k-m)^2}{m^2} + \frac{|k-m|}{m} + \frac{1}{m})\}. \, |\sin(k+1/2)t| \, dt]$$

$$= 0\ [\int_0^{\pi/m} \frac{|\Phi_x(t)|}{t} \sum_{m \le k \le m+m^\delta} \{\frac{1}{\sqrt{m}} \exp(-am^{-1}(k-m)^2)$$

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$$\left(\frac{(k-m)^2}{m^2}+\frac{|k-m|}{m}+\frac{1}{m}\right)\}.\ (|k-m|+m+1/2)t\,dt$$

$$+O\left[\int_{\pi/m}^{\pi}\frac{|\phi_x(t)|}{t}\sum_{m\le k\le m+m^{\delta}}\left\{\frac{1}{\sqrt{m}}\exp\left(-am^{-1}(k-m)^2\right)\right.\right.$$

$$\left.\left.\left(\frac{(k-m)^2}{m^2}+\frac{|k-m|}{m}+\frac{1}{m}\right)\right\}dt\right]$$

$$=O\left\{\sqrt{m}\int_0^{\pi/m}|\phi_x(t)|\,dt\right\}+0\left\{\frac{1}{\sqrt{m}}\int_{\pi/m}^{\pi}\frac{|\phi_x(t)|}{t}\,dt\right\}$$

(3.5) $\quad=O(m^{-\alpha}).$

Also in case where $k\le[q]<q<m$, we have

$$\frac{(k-m)}{\sqrt{m}}<\frac{(k-q)}{\sqrt{q}}<\frac{(k-[q])}{\sqrt{[q]}}<0.$$

Hence, from Ikeno [3, p. 260] following estimate results :

$$\left|\sqrt{\frac{a}{\pi q}}\exp\left(-aq^{-1}(k-q)^2\right)-\sqrt{\frac{a}{\pi m}}\exp\left(-am^{-1}(k-m)^2\right)\right|$$

$$=O\left\{\frac{1}{\sqrt{[q]}}\exp\left(-a[q]^{-1}(k-[q])^2\right)\left(\frac{(k-[q])^2}{[q]^2}+\frac{|k-[q]|}{[q]}+\frac{1}{[q]}\right)\right).$$

Proceeding as in the estimation of D_1 and using the above inequality we get

(3.6) $\quad|D_2|=O(m^{-\alpha}).$

Thus, Lemma 2 follows from (3.1) to (3.6).

4. PROOF OF THE THEOREM.

Since $S_k(f;x)-f(x)=\frac{1}{\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin t/2}\sin(k+1/2)t\,dt$

we have

$$t_p(f;x)-f(x)=\frac{1}{\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin t/2}\sum_{k=0}^{\infty}C_{pk}\sin(k+1/2)t\,dt+O(q^{-1/2})$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin t/2}\left[\left(\sum_{|k-q|\le q^{\delta}}+\sum_{|k-q|\le q^{\delta}}\right)C_{pk}\sin(k+1/2)t\right]dt+O(q^{-1/2})$$

(4.1) $\quad=S_1+S_2+0(q^{-1/2}).$

By (1.4) and the fact that $\sin t/2\ge t/\pi$ $(0<t\le\pi)$, we have

$$|S_2|\le\frac{|\phi_x(t)|}{t}\sum_{|k-q|>q^{\delta}}C_{pk}(k+1/2)t\,dt$$

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(4.2) $= O(\exp(-q^{\mu}))$.

Now making use of (1.3), we can write

$$S_1 = \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$

$$\{1 + O(\frac{|k-q|+1}{q}) + O(\frac{|k-q|^3}{q^2})\}. \sin(k+1/2)t\, dt$$

(4.3) $= S_3 + S_4 + S_5$.

We estimate S_4 as follows :

$$|S_4| \le \int_0^{\pi} \frac{|\phi_x(t)|}{t} \sum_{|k-q| \le q^{\delta}} \{\sqrt{\frac{a}{\pi q}} \exp -(aq^{-1}(k-q)^2)$$

$$O(\frac{|k-q|+1}{q})\} \; |\sin(k+1/2)t|\, dt$$

$$= \int_0^{\pi/q} \frac{|\phi_x(t)|}{t} \sum_{|k-q| \le q^{\delta}} \{\sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$

$$O(\frac{|k-q|+1}{q})\} \; (|k-q| + q + 1/2)t\, dt$$

$$+ \int_{\pi/q}^{\pi} \frac{|\phi_x(t)|}{t} \sum_{|k-q| \le q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2)$$

$$O(\frac{|k-q|+1}{q})\, dt$$

$$= O(\sqrt{q}\int_0^{\pi/q} t^{\alpha}\, dt)\, O(\frac{1}{\sqrt{q}}\int_{\pi/q}^{\pi} t^{\alpha-1}\, dt)$$

(4.4) $= O(q^{-\alpha})$.

Similarly,

(4.5) $|S_5| = O(q^{-\alpha})$.

Applying Lemma 1 and Lemma 2 and noting that $m = m(p)$ is an integer valued function of p we get

$$S_3 = \int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \exp(-\frac{mt^2}{4a}) \sin(m+1/2)t\, dt.$$

Now

$$|S_3| = O(1)\int_0^{\pi} t^{\alpha-1} \exp(-\frac{mt^2}{4a}) \; |\sin(m+1/2)t|\, dt$$

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$$= O(1) \int_0^{\pi/\sqrt{m}} t^{\alpha-1}\,dt + O(m^{-1}) \int_{\pi/\sqrt{m}}^{\pi} t^{\alpha-2} \frac{\partial}{\partial t}\left(\exp\left(-\frac{mt^2}{4a}\right)\right)dt$$

$$= O(m^{-\alpha/2}). \tag{4.6}$$

Collection of (4.1), (4.2) , ..., (4.6) completes the proof of the Theorem.

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